Regularization by sup-inf convolutions on Riemannian manifolds: an extension of Lasry-Lions theorem to manifolds of bounded curvature

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Introduction

For a (possibly infinite-dimensional) Riemannian manifold M, and a function $f : M \to \mathbb{R} \cup \{+\infty\}$, we define

$$f_\lambda(x) = \inf_{y \in M} \{f(y) + \frac{1}{2\lambda} d(x,y)^2\}$$

(well known in convex analysis as the Moreau-Yosida regularization of f in the case when M is a Hilbert space, and as a Lax-Oleinik semigroup in PDE theory).

Similarly, for a function $g : M \to \mathbb{R} \cup \{-\infty\}$ we define

$$g^\mu(x) = \sup_{y \in M} \{f(y) - \frac{1}{2\mu} d(x,y)^2\}.$$

Observe that $g^\mu = -(-g)_\mu$, and therefore all properties of functions of the form f_λ have an obvious analogue for functions of the form f^μ.

In 1986, J.-M. Lasry and P.-L. Lions proved that, if $M = E = \mathbb{R}^n$ or a Hilbert space, and if $f : E \to \mathbb{R}$ is bounded and uniformly continuous, then the functions

$$(f_\lambda)^\mu(x) = \sup_{z\in M}\inf_{y\in M}\{f(y) + \frac{1}{2\lambda}d(z,y)^2 - \frac{1}{2\mu}d(x,z)^2\}$$

are of class $C^{1,1}(E)$ and converge to f uniformly on E as $\lambda, \mu \to 0^+$ with $0 < \mu < \lambda/2$.

This is quite useful because of the fact that the correspondence

$$f(x) \mapsto (f_\lambda)^\mu(x) = \sup_{z\in M} \inf_{y\in M} \{f(y) + \frac{1}{2\lambda} d(z,y)^2 - \frac{1}{2\mu} d(x,z)^2\}$$

is explicit and preserves many significant geometrical properties that the given functions f may have, such as invariance by a set of isometries, infima, sets of minimizers, ordering, local or global Lipschitzness, and local or global convexity.

Lasry-Lions' regularization technique has also strong connections with PDE theory, through the Lax-Oleinik semigroup of a Hamilton-Jacobi equation. Indeed, the function $u(\lambda, x) = f_\lambda(x)$ is the unique viscosity solution of the equation

$$\frac{\partial u}{\partial \lambda} + \frac{1}{2}\|\nabla u\|^2 = 0$$

on $\mathbb{R}^+ \times E$, with initial data $u(0,x) = f(x)$.

Similarly, the function $v(\mu, x) = h^\mu(x)$ is the unique viscosity solution of

$$\frac{\partial v}{\partial \mu} - \frac{1}{2}\|\nabla v\|^2 = 0$$

on $\mathbb{R}^+ \times E$, with initial data $v(0,x) = h(x)$.

A. Fathi and P. Bernard have shown that the Lasry-Lions Theorem is true for compact Riemannian manifolds in a more general form (for Lax-Oleinik semigroups associated to a class of Hamilton-Jacobi equations).

However, the optimal Lipschitz constants of the gradients $\nabla(f_\lambda)^\mu$ do not seem to have been found. Fathi's and Bernard's proofs rely on compactness arguments that cannot be extended to noncompact manifolds.

Moreover, in the literature there does not seem to be a definition of global $C^{1,1}$ smoothness which makes sense for noncompact manifolds and has the usual properties that one should expect of such a notion.

What is a $C^{1,1}$ function?

If U is an open subset of $\mathbb{R}^n$ or a Hilbert space and $f : U \to \mathbb{R}$, saying that $f \in C^{1,1}(U)$ just means that $f \in C^1(U)$ and the gradient ∇f is a Lipschitz mapping from U into $\mathbb{R}^n$, that is, there exists $C \geq 0$ such that $\|\nabla f(x) - \nabla f(y)\| \leq C\|x - y\|$ for every $x, y \in U$. One says that C is a Lipschitz constant for ∇f, and the infimum of all such C is denoted by $\mathrm{Lip}(f)$.

The extension of this definition to the Riemannian setting is not an obvious matter, since for a C^1 function $f : M \to \mathbb{R}$ the vectors $\nabla f(x)$ and $\nabla f(y)$ belong to different fibres of TM and in general there is no global way to compare them that serves all purposes one may have in mind.

In his book on weak KAM theory, Fathi declares a function $f : M \to \mathbb{R}$ to be of class $C^{1,1}$ provided f is $C^{1,1}$ when looked at in charts. So does Bernard in his papers on the subject.

This definition is useful when M is compact, but does not lead to consistent results when M is not compact.

Fathi also introduced a pointwise Lipschitz constant of a gradient by means of a metric in the tangent bundle TM:

$$\mathrm{Lip}_x(\nabla f) = \limsup_{y,z \to x} \frac{d_{TM}(\nabla f(y), \nabla f(z))}{d_M(y,z)}.$$

One may then set $\mathrm{Lip}(\nabla f) = \sup_{x \in M} \mathrm{Lip}_x(\nabla f)$.

This leads to declaring a function $f \in C^1(M)$ to be of class $C^{1,1}(M)$ provided that the mapping $\nabla f : M \to TM$ is Lipschitz (with respect to the given metrics in M and TM). This notion of $C^{1,1}$ smoothness can be practical in some problems, but it has the disadvantage that $\mathrm{Lip}_x(f)$ is not finely controlled by the Hessian $D^2 f(x)$ when $f \in C^2(M)$.

For instance, for any Riemannian manifold M, if one endows TM with the Sasaki metric, since the parallel translation of the zero vector along a geodesic of M is always a geodesic in TM, one obtains, for every constant function c on M, that $\nabla c(x) = 0$ for every $x \in M$, hence also $D^2 c(x) = 0$ for every $x \in M$, and yet $\text{Lip}(\nabla c) = 1$.

Therefore, if one should define Lipschitzness of a gradient mapping ∇f by means of metrics in TM, then one would not be able to relate the Lipschitz constants of ∇f with the semiconvexity and semiconcavity constants of f.

Besides, metrics in the tangent bundle are difficult to handle and their geodesics are not well understood (except in simple manifols).

Let M be a Riemannian manifold (possibly infinite dimensional). For every $x_0 \in M$ there exists $R > 0$ such that the ball $B(x_0, 2R)$ is convex and $\exp_x : B_{T_xM}(0,R) \to B(0,R)$ is a C^∞ diffeomorphism for every $x \in B(x_0,R)$. If $x, y \in B(x_0,R)$, let us denote by $L_{xy} : T_xM \to T_yM$ the linear isometry between these tangent spaces provided by parallel translation of vectors along the unique minimizing geodesic connecting the points x and y.

When $i(M), c(M) > 0$, the isometry $L_{xy} : T_xM \to T_yM$ allows us to compare vectors (or covectors) which are in different fibers of TM (or T^*M), in a natural, semiglobal way.

Even when the global injectivity or convexity radii of M vanish, the following definition still makes sense.

Definition

Let M be a Riemannian manifold. We say that a function $f : M \to \mathbb{R}$ is of class $C^{1,1}(M)$ provided $f \in C^1(M)$ and there exists $C \geq 0$ such that for every $x_0 \in M$ there exists $r \in (0, \min\{i(x_0), c(x_0)\})$ such that

$$\|\nabla f(x) - L_{yx}\nabla f(y)\| \leq Cd(x,y)$$

for every $x, y \in B(x_0, r)$. We call C a Lipschitz constant of ∇f. We also say that ∇f is C-Lipschitz, and define $Lip(\nabla f)$ as the infimum of all such C.

When $i(M), c(M) > 0$, this definition is equivalent to the (apparently stronger) following one: $\|\nabla f(x) - L_{yx}\nabla f(y)\| \leq Cd(x,y)$ for every x, y with $d(x,y) < \min\{i(M), c(M)\}$.

As is well known in the Euclidean case, $C^{1,1}$ smoothness has much to do with semiconcavity and semiconvexity of functions, and in the general Riemannian setting we should also expect to find a strong connection between these notions.

Recall that a function $f : M \to \mathbb{R}$ is said to be convex provided $f \circ \gamma$ is convex on the interval $I \subseteq \mathbb{R}$ for every geodesic segment $\gamma : I \to M$. A function h is called concave if $-h$ is convex.

Definition

Let M be a Riemannian manifold. We say that a function $f : M \to (-\infty, +\infty]$ is (globally) semiconvex *if there exists $C > 0$ such that for every $x_0 \in M$ the function $M \ni x \mapsto f(x) + Cd(x, x_0)^2$ is convex.*

We say that f is locally semiconvex *if for every $x \in M$ there exists $r > 0$ such that $f_{|_{B(x,r)}} : B(x, r) \to [-\infty, +\infty]$ is semiconvex . If there exists $C \geq 0$ such that for every $x_0 \in M$ there exists $r > 0$ such that the function $B(x_0, r) \ni x \mapsto f(x) + Cd(x, y_0)^2$ is convex for every $y_0 \in B(x_0, r)$, then we will say that f is* locally C-semiconvex.

Finally, we say that $f : M \to [-\infty, +\infty]$ is uniformly locally C-semiconvex *provided that there exist numbers $C, R > 0$ such that for every $x_0 \in M$ the function*

$$B(x_0, R) \ni x \mapsto f(x) + Cd(x, x_0)^2$$

is convex. We will call C a constant of uniform local semiconvexity.

Theorem

Let M be finite dimensional, $f \in C^1(M, \mathbb{R})$, $C \geq 0$. TFAE:

1. *∇f is C-Lipschitz.*
2. *For every $x \in M$, $v \in T_xM$ with $\|v\| = 1$,*
$$\limsup_{t\to 0^+} \frac{1}{t} \|\nabla f(x) - L_{\exp_x(tv)x} \nabla f(\exp_x(tv))\| \leq C.$$
3. *For every $x_0 \in M$ and $\varepsilon > 0$ there exists $r > 0$ such that*
$$|f(\exp_x(v)) - f(x) - \langle \nabla f(x), v\rangle| \leq \frac{C + \varepsilon}{2} \|v\|^2$$
for every $x \in B(x_0, r)$ and $v \in B_{T_xM}(0, r)$.
4. *For every $C' > C$, f is locally $\frac{C'}{2}$-semiconvex and locally $\frac{C'}{2}$-semiconcave.*
5. *For every $x \in M$ and every $\varepsilon > 0$ there exists $r > 0$ such that, if $F := f \circ \exp_x : B(0, r) \to \mathbb{R}$, then, for every $u, v \in B_{T_xM}(0, r)$,*
$$\|\nabla F(u) - \nabla F(v)\| \leq (C + \varepsilon)\|u - v\|.$$

Moreover, if $f \in C^2(M, \mathbb{R})$ then any of the above statements is also equivalent to the following estimate for the Hessian of f:

(6) $\|D^2 f\| \leq C$.

Finally, if M is of bounded sectional curvature with $i(M), c(M) > 0$, any of the conditions $(1) - (5)$ is equivalent to

(4') For every $C' > C$ the function f is uniformly locally $\frac{C'}{2}$-semiconvex and uniformly locally $\frac{C'}{2}$-semiconcave,

and also to

(1') There exists $R > 0$ such that for every $x_0 \in M$ we have

$$\|L_{yx}(\nabla f(y)) - \nabla f(x)\| \leq C d(x, y)$$

for every $x, y \in B(x_0, R)$.

Finally, when M is infinite dimensional the implications $(1) \iff (2) \implies (3) \implies (4)$ and $(5) \implies (1)$ remain true, and any of these conditions is equivalent to (6) if $f \in C^2(M)$. We also have that (4) implies that f is $C^{1,1}(M)$ with $\text{Lip}(\nabla f) \leq 6C'$.

The main result

Theorem

Let M be a Riemannian manifold (possibly infinite dimensional) with sectional curvature K such that $-K_0 \leq K \leq K_0$, and such that $i(M), c(M) > 0$ respectively. Let $f : M \to \mathbb{R}$ be uniformly continuous and bounded, and $q > 1$. Then there exists $\lambda_0 = \lambda(K_0, q, f) > 0$ such that for every $\lambda \in (0, \lambda_0]$ and every $\mu \in (0, \lambda/2q]$ the regularizations $(f_\lambda)^\mu$ are uniformly locally $\frac{q}{2\mu}$-semiconvex and uniformly locally $\frac{q}{2\mu}$-semiconcave, and they converge to f, uniformly on M, as $\lambda, \mu \to 0$.
In particular we have that $(f_\lambda)^\mu \in C^{1,1}(M)$ for every such λ, μ. Moreover, we have the following estimations of the Lipschitz constants of $\nabla\left((f_\lambda)^\mu\right)$:

$$Lip\left(\nabla\left((f_\lambda)^\mu\right)\right) \leq \frac{q}{\mu} \text{ if } M \text{ is finite dimensional, and}$$

$$Lip\left(\nabla\left((f_\lambda)^\mu\right)\right) \leq 6\frac{q}{\mu} \text{ if } M \text{ is infinite dimensional.}$$

Finally, if f is Lipchitz then so is $(f_\lambda)^\mu$, and we have

$$\lim_{\lambda,\mu \to 0^+} Lip\left((f_\lambda)^\mu\right) = Lip(f).$$

If one drops the assumption that f is bounded, then the result fails in general (even in the hiperbolic plane).

We also have counterexamples showing that one cannot dispense with the assumption that the sectional curvature K be bounded.

Some ingredients of the proofs

Besides parallel translation, another natural, semiglobal way to compare vectors in different fibers T_xM, T_yM of TM with $d(x,y) < i(x)$ is by means of the differential of the exponential map

$$d\exp_x(v) : T(T_xM)_v \equiv T_xM \to T_yM,$$

where $v = \exp_x^{-1}(y)$.

It is a straightforward consequence of the definition of P as a solution to a linear ordinary differential equation with initial condition $P(0) = h$, and of the fact that $d\exp_x(0)(h) = h$, that

$$\lim_{y\to x} \sup_{h\in T_xM, \|h\|=1} \left| d\exp_x\left(\exp_x^{-1}(y)\right)(h) - L_{xy}(h)\right| = 0.$$

That is, $\lim_{y\to x} \|d\exp_x\left(\exp_x^{-1}(y)\right) - L_{xy}\|_{\mathcal{L}(T_xM, T_yM)} = 0.$

However, we need much sharper estimations on the rate of this convergence. In particular, we need to use the that, locally, one has

$$\|d\exp_x\left(\exp_x^{-1}(y)\right) - L_{xy}\|_{\mathcal{L}(T_xM,T_yM)} = O\left(d(x,y)^2\right).$$

There are well known estimates of the form

$$d\exp_x\left(t\frac{v}{\|v\|}\right)(th) - P(th) = O(t^3),$$

but we need (and show) this kind of estimate to hold locally uniformly with respect to x, v, h. As a consequence we also show that

$$\|d(\exp_x^{-1})(y) \circ L_{xy} - I\|_{\mathcal{L}(T_xM,T_xM)} = O\left(d(x,y)^2\right)$$

locally uniformly.

If M is a Riemannian manifold of nonpositive sectional curvature K with $i(M) > 0$, $c(M) > 0$, it is well known that the functions $B(x_0, R) \times B(x_0, R) \ni (x, y) \mapsto d(x, y)^2$ and $B(x_0, R) \ni x \mapsto d(x, x_0)^2$ are C^∞ and convex, provided that $2R < \min\{i(M), c(M)\}$.

On the other hand, in the general case (for instance if $K > 0$) it is not true that the mapping $(x, y) \mapsto d(x, y)$ is locally convex, not even when (x, y) move in an arbitrarily small neighborhood of a point $(x_0, x_0) \in M \times M$.

In this situation it is somewhat surprising that we still have this compensation property:

Lemma

Let M be a Riemannian manifold with sectional curvature K such that $-K_0 \leq K \leq K_0$ for some $K_0 > 0$. Assume also that $i(M) > 0$ and $c(M) > 0$. Let $q > 1$. Then there exists $R = R(K_0, q) > 0$ such that for every $C \geq 0$, for every $A \geq 2C$ and $B \geq qA$, and for every $x_0 \in M$ and $y_0 \in B(x_0, R)$, the function

$$\varphi(x, y) := Ad(x, y)^2 + Bd(x, x_0)^2 - Cd(y, y_0)^2$$

is convex on $B(x_0, R) \times B(x_0, R)$.