On Random Unconditional Convergence in rearrangement invariant spaces

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Authorship

Joint work with:

Authorship

Joint work with:

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Outline

1. Random Unconditional Convergence

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2. RUC in functions spaces

3. An extension of a result by DSS

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- Recall: a series $\sum x_n$ in a Banach space X *converges unconditionally* if for every reordering $\sigma : \mathbb{N} \to \mathbb{N}$ the series

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  (c) *Bounded multiplier convergence*, if for every scalars $|a_n| \leq M$ the series

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  for almost every choice of signs $\varepsilon_i = \pm 1$, and all $x \in [x_i]$.

- Equivalently: there exists a constant $K > 0$ such that

$$\int_0^1 \Big\| \sum_{i=1}^{n} c_i r_i(t) x_i \Big\|_X dt \leq K \Big\| \sum_{i=1}^{n} c_i x_i \Big\|_X,$$

  for every $n = 1, 2, \dots$ and arbitrary scalars $c_1, c_2, \dots, c_n$. Here, (r_n) are the Rademacher functions.

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- Let (x_i, x_i^*) be a biorthogonal, fundamental and total system in X. Then:

  (x_i) is an unconditional basis in X iff (x_i, x_i^*) and (x_i^*, x_i) are RUC systems in X and X^*, resp.

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- $L^1([0,1])$ has no fundamental RUC system.

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Here: G denotes the closure of L_∞ in the Orlicz space generated by the function $\exp(t^2) - 1$.

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- The following questions naturally arise:
  - To what extend the uniform boundedness (in the L_∞-norm) is relevant for the existence of RUC systems?
  - Can uniform boundedness in the L_∞-norm be replaced by the uniform boundedness of the system in a larger space (i.e., for a weaker norm)?

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- Examples: $L^{p,\infty}$, the Orlicz space L_M generated by $M(t) := e^{t^2} - 1$.

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- For this, we proceed to estimate, in the average, the distribution of the signed series

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$$\int_0^1 \left|\left\{s \in [0,1] : \left|\sum_{i=1}^{\infty} c_i r_i(t) f_i(s)\right| \geq \tau\right\}\right| dt \leq 2 \int_0^1 e^{(-K\tau^2\psi(t)^2)}\, dt,$$

for all $\tau > 0$.

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   That is, $M(\varphi_\beta)_0 \subset X$ is more restrictive than $G \subset X$.

Thank you

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