Nonexistence of certain universal polynomials between Banach spaces

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In honor of Richard M. Aron

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Universal operators

Theorem (Lindenstrauss, Pełczyński, 1968)

Let $\sigma : \ell_1 \to \ell_\infty$ be the *sum* operator defined by

$$\sigma(x) = \left(\sum_{i=1}^{n} x_i \right)_{n=1}^{\infty} \qquad \text{for } x = (x_n)_{n=1}^{\infty} \in \ell_1 \,.$$

Then, an operator $T \in \mathcal{L}(X, Y)$ is not weakly compact if and only if there exist operators A and B such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle A}\big\uparrow & & \big\downarrow{\scriptstyle B} \\ \ell_1 & \xrightarrow[\sigma]{} & \ell_\infty \end{array}$$

Universal operators

We say that σ is a *universal* non-weakly compact operator.

Theorem (Johnson, 1971)

The formal identity operator $J : \ell_1 \to \ell_\infty$ is a universal non-compact operator.

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Preliminaries

An operator ideal $\mathcal{U}$ is *surjective* if, given $T \in \mathcal{L}(E, F)$ and a surjective operator $q : G \to E$, we have that $T \in \mathcal{U}$ whenever $Tq \in \mathcal{U}$.

Proposition (Aron, Schottenloher,1976)

Given Banach spaces X and Y, and $k \in \mathbb{N}$, the space $\mathcal{P}({}^kX, Y)$ is isomorphic to a complemented subspace of $\mathcal{P}({}^{k+1}X, Y)$.

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Idea of proof:

Given $\psi \in S_{X^*}$, choose $x_0 \in X$ so that $\psi(x_0) = 1$. Define the operators

$$\mathcal{P}({}^kX, Y) \xrightarrow{\ j\ } \mathcal{P}({}^{k+1}X, Y) \xrightarrow{\ \pi\ } \mathcal{P}({}^kX, Y)$$

by $j(R)(x) := \psi(x)R(x)$ for all $R \in \mathcal{P}({}^kX, Y)$ and $x \in X$, and

$$\pi(P)(x) := \sum_{i=1}^{k+1} \binom{k+1}{i} \psi(x)^{i-1}(-1)^{i-1}\widehat{P}\left(x_0^i, x^{k+1-i}\right) =: Q(x)$$

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Preliminaries

Let $\mathcal{U}$ be a closed surjective operator ideal. We denote by $\mathcal{C}_{\mathcal{U}}(E)$ the collection of all sets $A \subset E$ so that $A \subseteq T(B_Z)$ for some Banach space Z and some operator $T \in \mathcal{U}(Z, E)$.

Definition

Given a closed surjective operator ideal $\mathcal{U}$ and an integer $k \geq 1$, let

$$\mathcal{P}_{\mathcal{U}}(^kX, Y) := \left\{ P \in \mathcal{P}(^kX, Y) : P(B_X) \in \mathcal{C}_{\mathcal{U}}(Y) \right\}.$$

The space $\mathcal{P}_{\mathcal{U}}(^kX, Y)$ will be endowed with the supremum norm.

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Preliminaries

Proposition

If $\mathcal{U}$ is a closed surjective operator ideal, restricting j and π to the spaces $\mathcal{P}_{\mathcal{U}}$, we have

$$\mathcal{P}_{\mathcal{U}}(^kX, Y) \xrightarrow{\ j\ } \mathcal{P}_{\mathcal{U}}(^{k+1}X, Y) \xrightarrow{\ \pi\ } \mathcal{P}_{\mathcal{U}}(^kX, Y)\,,$$

that is, j and π take polynomials in $\mathcal{P}_{\mathcal{U}}$ into polynomials in $\mathcal{P}_{\mathcal{U}}$.

Universal non-compact polynomial

Lemma

Given $k \in \mathbb{N}$ ($k \geq 1$), if there is a universal k-homogeneous non-compact polynomial, then there is a universal k-homogeneous non-compact polynomial defined on ℓ_1.

Universal non-compact polynomial

Proposition

Let $k \in \mathbb{N}$ ($k \geq 2$). If there is a universal k-homogeneous non-compact polynomial $P_0 \in \mathcal{P}({}^k\ell_1, F)$, it can be taken of the form $P_0 = \xi^{k-1}J$, where $\xi \in \ell_\infty$.

Theorem

Let $k \geq 2$ be an integer. Then there is no universal k-homogeneous non-compact polynomial.

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Universal non-unconditionally converging operator

Proposition

Let $T \in \mathcal{L}(c_0, F)\backslash\mathcal{K}(c_0, F)$. Then there are operators $A \in \mathcal{L}(c_0, c_0)$ and $B \in \mathcal{L}(F, \ell_\infty)$ such that $I = B \circ T \circ A$, where $I : c_0 \hookrightarrow \ell_\infty$ is the natural embedding. If F is separable, I may be taken to be the identity map on c_0.

Corollary

The natural embedding $I \in \mathcal{L}(c_0, \ell_\infty)$ is universal for the class of non-unconditionally converging operators.

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Universal non-unconditionally converging polynomial

Lemma

Given a polynomial $P \in \mathcal{P}(^kE, F) \setminus \mathcal{P}_{\mathrm{uc}}(^kE, F)$, where $k \geq 1$ is an integer, there is an operator $j : c_0 \to E$ such that $P \circ j \in \mathcal{P}(^kc_0, F) \setminus \mathcal{P}_{\mathrm{uc}}(^kc_0, F)$.

Proposition

Given an integer $k \geq 2$, if there is a universal non-unconditionally converging k-homogeneous polynomial P_0, it may be taken of the form $P_0 := \xi^{k-1} I$, where $\xi \in \ell_1$ and $I \in \mathcal{L}(c_0, \ell_\infty)$ is the natural embedding.

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Proposition

Given an integer $k \geq 2$, if there is a universal non-unconditionally converging k-homogeneous polynomial, it may be chosen of the form $(e_1^*)^{k-1}I$, where $I \in \mathcal{L}(c_0, \ell_\infty)$ is the natural embedding.

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Proposition

Given an integer $k \geq 2$, assume there is a universal non-unconditionally converging k-homogeneous polynomial which, by Proposition **??**, may be taken of the form $(e_1^*)^{k-1} I$. Given a polynomial $P \in \mathcal{P}(^k c_0, c_0) \backslash \mathcal{P}_{\mathrm{uc}}(^k c_0, c_0)$, let $A \in \mathcal{L}(c_0, c_0)$ and $B \in \mathcal{L}(c_0, \ell_\infty)$ be operators such that $(e_1^*)^{k-1} I = BPA$. Then A is an into isomorphism.

Theorem

Given an integer $k \geq 2$, there is no universal non-unconditionally converging k-homogeneous polynomial.

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