Information and σ-algebras

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XIII Encuentro de Análisis Funcional Murcia-Valencia
in honor of 70 Birthday of Prof. Richard M. Aron,
Valencia, December 13th, 2014

Abstract

This work is an attempt to clarify the meaning of the information that an agent receives from a signal, from an experiment or from her own ability to precise the true state of nature that already occurs.

Motivation

Given a signal, or equivalently, a partition of the set of states of nature, it is frequently argued in the literature that technical reasons lead to consider the σ-algebra generated by the partition as the informational content of the signal.

However, Billingsley, ("Probability and Measure", Wiley, 1995), argued that the interpretation **of σ-algebras as information is weak**. Billingsley's argument is concerned with the fact that, sometimes, the σ-algebra generated by the informational partition does not corresponds to the heuristic equating of information.

Motivation

Later, J. Dubra and F. Echenique, precise Billingsley's objection and pointed out in their paper "Information is not about measurability" (Mathematical Social Sciences, 2004), that the use of σ-algebras as the informational content of a signal or a partition is **inadequate**.

Dubra and Echenique show by a simple example that the use of the σ-algebra generated by a partition, as a model of information, leads to a paradoxical conclusion: a decision-maker prefers less information than more. This comes from the fact that finer partitions may not generate finer σ-algebras.

Main objectives

The objectives of this work are:

1. Give a precise definition of the informational content of a partition (main definition).
2. Study the properties of the informational σ-algebra.
3. Show that, with this main definition of informational content of a partition, Billingsley and Dubra-Echenique's concerns are no longer a problem.
4. Establish a Backwell type theorem.

Order

This presentation is structured as follows:

1. Background: Formal definitions (partitions and signals)
2. Dubra and Echenique's example and their conclusions
3. Our main definition and its interpretation
4. The σ-algebra of events
5. The case of finite or countable partitions
6. Signals and experiments
7. Backwell type theorem

Partitions

Ω is the set (finite or infinite) of states of nature

The set of subsets of Ω is $\mathcal{P}(\Omega) := \{A; A \subset \Omega\}$

A partition of Ω is a family of subsets of $\Omega, \tau \subset \mathcal{P}(\Omega)$, such that

1. $\bigcup_{X \in \tau} X = \Omega$;
2. If $X, Y \in \tau$ and $X \neq Y$ then $X \cap Y = \emptyset$.

If τ and τ' are partitions of Ω, τ' is finer than τ if every element of τ is a union of elements of τ'. In this case we write $\tau' \geq \tau$. Formally, $\tau' \geq \tau$ if for all $X \in \tau$, and for all $z \in X$ there exists $Y \in \tau'$, $Y \subset X$ with $z \in Y$.

Signals

A signal on Ω with images on S is any mapping $f : \Omega \to S$

The signal f induces a partition on Ω, defined by
$\tau_f = \{f^{-1}(s), s \in S\}$

Reciprocally, any partition τ can be identified with a signal. For it, denote by $\sim$ the equivalence relation defined by $\omega \sim \omega'$ if and only if $\tau(\omega) = \tau(\omega')$, where $\tau(z)$ denotes the unique element (block) of τ containing z. Let $\frac{\Omega}{\sim}$ the quotient set; that is $\tau = \frac{\Omega}{\sim}$ and define

$$f : \Omega \to \tau = \frac{\Omega}{\sim}$$

as the natural projection $f(\omega) = \tau(\omega)$
It is clear that the partition induced by the signal f is, precisely, $\tau_f = \tau$

Dubra-Echenique example

Let the state of the world be a real number between 0 and 1, so the set of possible states is $\Omega = [0, 1]$.

Suppose that a decision-maker can choose either to be perfectly informed, (she gets to know the exact value of $\omega \in \Omega$), or only be told if the true state ω is smaller or larger than $1/2$.

In the first case, the information is represented by the partition τ of all elements of Ω;

$\tau = \{\{\omega\}, \omega \in \Omega\}$.

In the second case, the information is represented by the partition

$\tau' = \{[0, \frac{1}{2}); [\frac{1}{2}, 1]\}$.

Example

Let denote by $\sigma(\tau)$ and $\sigma(\tau')$, the σ-algebras generated by τ and τ' respectively. It is easy to see that:
$\sigma(\tau') = \{\emptyset; \Omega; [0, \frac{1}{2}), [\frac{1}{2}, 1]\}$
while $\sigma(\tau)$ is the collection of sets in [0, 1] that are either countable or have countable complement.

Observe that while τ is finer than τ', the σ-algebras $\sigma(\tau)$ and $\sigma(\tau')$ are not comparable.

Moreover, in spite that τ is the full information, $\sigma(\tau)$ is not informative at all.

D-E Conclusions

The conclusions obtained from the example are clear:

1. Finer partitions do not necessarily generate finer algebras or σ-algebras.
2. The example allows to construct a family of numerical examples of decision-makers that can use, as information, either $\sigma(\tau)$ or $\sigma(\tau')$, to conclude that the decision-maker strictly prefers $\sigma(\tau')$ over $\sigma(\tau)$.

   That is, the σ-algebra generated by the full information could be strictly less preferred (by a decision-maker) than the σ-algebra generated by the poor information.

D-E Conclusions

Consequently, Dubra and Echenique write:

"We do not argue that using σ-algebras as the informational content of signals (partitions) is always inappropriate. We only want to emphasize that one should be careful when using σ-algebras as the informational content of signals" (or partitions).

Main definition

Consider a signal $f : \Omega \to S$ or the corresponding partition τ_f of Ω
A set $A \subseteq \Omega$ is an informed set (or an **event**, in relation with the information given by τ_f) if and only if

$$\forall X \in \tau, X \subset A \text{ or } X \subset A^c$$

Equivalently, A is an **event**, if and only if, for every $s \in S$,

$$f^{-1}(s) \subset A \text{ or } f^{-1}(s) \subset A^c$$

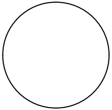
Interpretation

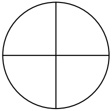
An **event**, or informed set, has a very natural meaning. An **event** is a set, $A \subset \Omega$, such that for every $X \in \tau_f$, if X occurs then necessarily A or necessarily A^c occurs.

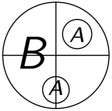
On the other hand, A is not an **event**, if and only if, there exist $\omega \in A$ and $\omega' \in A^c$ such that $f(\omega) = f(\omega')$.

If ω occurs, the decision-maker receives the image of the signal $f(\omega) = f(\omega')$, that corresponds to ω and also to ω' and she does not know if A occurs or not. Consequently, A is not an event.

Examples

$\Omega =$ 

$\tau =$ 



A is not an event

B is an event

Examples II

Let denote by $\mathcal{I}(\tau_f)$ the family of **events** or sets informed by the partition τ_f, or equivalently the family of sets informed by the signal f.

In the example of D-E, $\Omega = [0, 1]$

If $\tau = \{\{\omega\}, \omega \in \Omega\}$, then

$$\mathcal{I}(\tau) = \mathcal{P}(\Omega) := \{A; A \subset \Omega\} \neq \sigma(\tau)$$

If $\tau' = \{[0, \frac{1}{2}); [\frac{1}{2}, 1]\}$

$$\mathcal{I}(\tau') = \{\emptyset; \Omega; [0, 1/2); [1/2, 1]\} = \sigma(\tau')$$

The σ-algebra of events

Proposition. *The family of events $\mathcal{I}(\tau_f)$ is a σ-algebra that contains τ_f.*

First, note that since τ_f is a partition, $\tau_f \subset \mathcal{I}(\tau_f)$. The definition of $\mathcal{I}(\tau_f)$ is symmetric in (A, A^c). Thus, if $A \in \mathcal{I}(\tau_f)$ then $A^c \in \mathcal{I}(\tau_f)$. Suppose now that $A_i \in \mathcal{I}(\tau_f)$ for every i. Let $A = \cup_i A_i$. Suppose $X \in \tau_f$. If for some i, $X \subset A_i$ then $X \subset A$. If for every i it is not true that $X \subset A_i$ then $X \subset A_i^c$ for every i. Thus, $X \subset \cap_i A_i^c = A^c$. Hence, $A \in \mathcal{I}(\tau_f)$.

Remark that we already show that $\mathcal{I}(\tau_f)$ is closed for uncountable unions.

The σ-algebra of events

Our main point is to emphasize that the informational content of a signal f or equivalently of partition τ_f is, precisely, the σ-algebra $\mathcal{I}(\tau_f)$.

Proposition. $\tau \geq \tau'$ if and only if $\mathcal{I}(\tau') \subset \mathcal{I}(\tau)$. To have a finer partitions is equivalent to have more information.

This makes clear that our interpretation of information solves the concern set by D-E.

Proof

$\tau' \leq \tau$ implies $\mathcal{I}(\tau') \subseteq \mathcal{I}(\tau)$.

Let $A \in \mathcal{I}(\tau')$ and $Y \in \tau$. There exists $X \in \tau'$ such that $Y \subseteq X$. Then either $X \subseteq A$ and thus $Y \subseteq A$ or $X \subseteq A^c$ and thus $Y \subseteq A^c$.

$\mathcal{I}(\tau') \subseteq \mathcal{I}(\tau)$ implies $\tau' \leq \tau$.

Let $X \in \tau'$ and $z \in X$. Let us consider the unique element $Y \in \tau$ such that $z \in Y$. As $X \in \mathcal{I}(\tau)$ then $Y \subseteq X$ (since $Y \subset X^c$ is impossible).

Finite or Countable partitions

Proposition

If τ is finite or countable then $\mathcal{I}(\tau) = \sigma(\tau)$.

Proof

Let $\tau = \{X_j; j \in \mathbb{N}\}$ be a countable partition of Ω. It is immediate that $\mathcal{I}(\tau) \supset \sigma(\tau)$.
Let $A \in \mathcal{I}(\tau)$. Let $J = \{i \in \mathbb{N}; X_j \subset A\}$. Thus $\cup_{j \in J} X_j \subset A$. Since $A \in \mathcal{I}(\tau)$ for every $j \notin J$, $X_j \subset A^c$ and therefore $\cup_{j \in J} X_j = A$.
Since J is countable $A \in \sigma(\tau)$.

Signals and experiments

Consider that the set of states is a measurable space $(\Omega, \mathcal{F})$.

A signal f on Ω with images on $(S, \mathcal{B})$ is measurable iff $f^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$.

The σ-algebra generated by f, denoted by $\sigma(f, \mathcal{B})$, is the smallest σ-algebra on Ω for which f is measurable. We say that a σ-algebra $\mathcal{B}$ on S distinguishes f if $\{s\} \in \mathcal{B}$, for all $s \in S$.

Without loss of generality, we can assume that $f(\Omega) = S$.

Proposition. $\mathcal{I}(P_f) = \sigma(f, \mathcal{P}(S))$

Signals and experiments

Suppose $f : \Omega \to Y$ and $g : \Omega \to Z$ are signals on Ω and $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ are measurable spaces.

An experiment on $(Y, \mathcal{B})$ is a collection $\alpha = (m_\omega)_{\omega \in \Omega}$ of probability measures on $(Y, \mathcal{B})$.

Notice that an experiment is just a function from Ω to the set of probability measures on some space $(Y, \mathcal{B})$.

In fact a classical signal $f : \Omega \to Y$ can be identify with the experiment that associates with each ω the probability degenerated in $f(\omega)$.

Blackwell type theorem

Following Dubra and Echenique, let C denote the set of consequences.

An act is a function $a : \Omega \to C$, $A = C^{\Omega}$ is the set of acts.

A decisionmaker is a complete, transitive, binary relation $\succsim$ on A.

A decisionmaker gets her information from signals $f : \Omega \to Y_f$ for some space Y_f.
The decisionmaker is informed of the value taken by f and she must then choose a consequence in C.
An act $a : \Omega \to C$ is $f - \mathit{feasible}$ if $a(\omega) = a(\omega')$ whenever $f(\omega) = f(\omega')$

A decisionmaker $\succsim$ prefers signal f to g if and only if, for any $g - \mathit{feasible}$ act a, there exists an $f - \mathit{feasible}$ act $\hat{a}$ such that $\hat{a} \succsim a$.

Blackwell type theorem

Let $f : \Omega \to Y_f$ and $g : \Omega \to Y_g$ two signals. The following statements are equivalent:

- A desicionmaker prefers the signal (partition) f to g.
- $\mathcal{I}(\tau_g) \subseteq \mathcal{I}(\tau_f)$.
- There exits $h : Y_f \to Y_g$ such that $g = h \circ f$.

Sigma-algebras as information

In scholarly practice, we do not have a partition ready for use. Thus, we consider the scenario where the starting points are σ-algebras, instead of signals or partitions.
First, we examine the case of a countably generated σ-algebra. For it we require some ingredients:

- Polish spaces, Analytic sets, Blackwell σ-algebras, strongly Blackwell σ-algebras
- (Boreleans σ-algebra of a Polish space is a strongly Blackwell σ-algebra)
- Let be $\mathcal{A}$ strongly Blackwell σ-algebra and $\mathcal{G}$ a countably generated sub-σ-algebra, then $\mathcal{I}(\text{atoms } \mathcal{G}) \cap \mathcal{A} = \mathcal{G}$

Partitions from general Sigma-algebras

Consider a fixed probability space $(\Omega, \mathcal{A}, P)$, with $\mathcal{A}$ strongly Blackwell σ-algebra.

The sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are equivalents iff:
for all $B \in \mathcal{B}$ there is a $C \in \mathcal{C}$ such that $P(B \Delta C) = 0$ and
for all $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ such that $P(C \Delta B) = 0$.

Lemma (Stinchcombe, 1990) Every sub-σ-algebra of $\mathcal{A}$ is equivalent to a countably generated sub-σ-algebra of $\mathcal{A}$.

Partitions from general Sigma-algebras

Our last theorem states, in an informal sense, that information and measurability are equivalent as long as the information is suitably defined through equivalent countably generated σ-algebras.

Theorem

Suppose $\mathcal{B}$ and $\mathcal{C}$ are σ-algebras contained in $\mathcal{A}$. Then, except for the removal of a null subset of Ω, $\mathcal{B} \subset \mathcal{C}$ if and only if the partition of the atoms of $\mathcal{C}$ is finer than the partition of the atoms of $\mathcal{B}$.

Example

The σ-algebra in Billingsley and in Dubra and Echenique's example, $\mathcal{G}$ =$\{A : A$ countable or A^c countable$\}$ is not countably generated. The σ-algebra $\{\emptyset; \Omega\}$ is equivalent to $\mathcal{G}$ and therefore the partition of $\mathcal{G}$ is not the singletons partition (i.e., full information) but rather the coarsest partition $\tau(\mathcal{G}) = \{\Omega\}$

Conclusions

- We have given a precise and natural definition of the informational content of a signal. Our first conclusion is that the fact of considering σ-algebras to model the informational content of a signal is not due to technical reasons; the family of informed sets is itself a σ-algebra.
- Our results validate the use of the σ-algebra generated by the partition or the informational content of a signal, in the case of finite or countable partitions, as it is the case of several articles, and in particular of the papers quoted by Dubra and Echenique.

Conclusions

- The main result in this paper is that finer partitions generate finer σ-algebras of informed sets, and conversely, finer σ-algebras of informed sets come from finer partitions. This provides a formal and solid basis to the heuristics related to the informational content of a signal.
- Finally our last conclusion is that, as a consequence of our results, the concerns set by Billingsley and by Dubra and Echenique have a conceptually satisfactory explanation.

Thank you for your attention.

RICHARD, Many Thanks and

CONGRATULATIONS !