Fixed Point Property in Banach spaces and some connections with Renorming Theory.

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Dedicated to Richard Aron
on the occasion of his seventieth birthday

Introduction

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X.

Definition

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

First positive results

Theorem (D. Göhde, F. Browder, 1965)

Let X be a uniformly convex Banach space. Let $C \subset X$ be a closed convex bounded subset and $T : C \to C$ is nonexpansive. Then T has a fixed point.

Definition

A Banach space is said to have normal structure (NS) is for every closed convex bounded subset C with $diam(C) > 0$, there exists some $x_0 \in C$ such that

$$\sup_{y \in C} \|x_0 - y\| < diam(C).$$

Theorem (A. Kirk, 1965)

Let X be a reflexive Banach space with normal structure. Let $C \subset X$ be a closed convex bounded subset and $T : C \to C$ nonexpansive. Then T has a fixed point.

Uniformly convex spaces are reflexive and have normal structure.

Definition

A Banach space X has the fixed point property (FPP) if every **non-expansive** mapping $T : C \to C$, where C is a **closed convex bounded** subset of X, has a fixed point.

Hilbert spaces, uniformly convex Banach spaces or, more generally, reflexive Banach spaces with normal structure have the FPP.

Open problem

Does reflexivity imply FPP?

The FPP is an isometric property.

The FPP is not preserved by isomorphisms:

The nonexpansiveness of the mapping ($\|Tx - Ty\| \leq \|x - y\|$) depends on the underlying norm.

Open problem

If X is isomorphic to a uniformly convex Banach space (superreflexive), does X satisfies the FPP?

Given X a Hilbert space, does there exist some equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ fails to have the FPP?

Stability of the FPP

Given X, Y two isomorphic Banach spaces, the Banach-Mazur distance is defined as

$$d(X, Y) = \inf\{\ \|T\|\ \|T^{-1}\|;\ T : X \to Y \text{ isomorphism}\}$$

Given a Banach space X with the FPP.
Does there exists some $k = k(X)$ such that a Banach space Y has the FPP whenever

$$d(X, Y) < k?$$

- $1 < p < +\infty$: If

$$d(\ell_p, Y) < \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}$$

then Y satisfies the FPP.

- Let H be a Hilbert space. If

$$d(H, Y) < \sqrt{\frac{5+\sqrt{17}}{2}} = 2.13578...$$

then Y has the FPP (E. Mazcuñán Navarro, 2005).

The failure of the FPP

There exist some Banach spaces failing the FPP, such us $(\ell_1, \|\cdot\|_1)$ and $(c_0, \|\cdot\|_\infty)$.

Every Banach space which contains an isometric copy of either ℓ_1 or c_0 fail to have the FPP.

The failure of the FPP in ℓ_1

Let $C = \overline{co}\{e_n\}_n = \left\{ \sum_{n=1}^{\infty} t_n e_n : 0 \leq t_n \leq 1, \sum_{n=1}^{\infty} t_n = 1 \right\}$. Define $T : C \to C$ by

$$T\left(\sum_{n=1}^{\infty} t_n e_n \right) = \sum_{n=1}^{\infty} t_n e_{n+1}$$

T is nonexpansive a fixed point free.

Question

If a Banach space X fails to have the FPP, does there exist an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ does have the FPP?

Could either ℓ_1 or c_0 be FPP-renormable?

There exist some Banach spaces which are not FPP-renormable

First concepts

Definition (J. Hagler, 1972)

A Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of ℓ_1 (a.i.c. of ℓ_1) if there exist $(x_n) \subset X$ and $(\epsilon_n) \downarrow 0$ such that

$$\sum_n (1-\epsilon_n)|t_n| \leq \left\| \sum_n t_n x_n \right\| \leq \sum_n |t_n|$$

for all $(t_n) \in \ell_1$.

Theorem (P. Dowling, C. Lennard, B. Turett, 1996)

If a Banach space X contains an a.i.c. of ℓ_1, then X fails to have the FPP.

Let Γ be an uncountable set. Every renorming of $\ell_1(\Gamma)$ contains an asymptotically isometric copy of ℓ_1.

Theorem

$\ell_1(\Gamma)$ cannot be renormed to have the FPP.

Theorem

If X is a Banach space which contains an isomorphic copy of ℓ_1, then X^* contains a copy of $\ell_1(\Gamma)$ for some uncountable Γ. In particular X^* cannot be renormed to have the FPP. The space ℓ_∞ is not FPP-renormable.

Theorem (P. Dowling, C. Lennard, B. Turett)

For Γ uncountable, $c_0(\Gamma)$ is not FPP-renormable.

Positive results

Theorem (T. Domínguez-Benavides, 2008)

Every reflexive Banach space admits an equivalent norm with the FPP.

Idea of the proof

X has a Markushevich basis, then there exists a one-to-one continuous embedding

$$J : X \to c_0(\Gamma).$$

Take the norm:

$$|x|^2 := \|x\|^2 + \|Jx\|_\infty^2$$

$(X, |\cdot|)$ has the FPP.

ℓ_1 is FPP-renormable

Some background

- Since 1965, it was conjectured that FPP could imply reflexivity.
- By James' theorem, if $|\cdot|$ is an equivalent norm in ℓ_1, then $(\ell_1, |\cdot|)$ contains a subspace which is "almost" isometric to $(\ell_1, \|\cdot\|_1)$. Then, for a long time it was thought that for every equivalent norm $|\cdot|$ in ℓ_1, the space $(\ell_1, |\cdot|)$ could fail the FPP.

Lemma (P. Dowling, C. Lennard, B. Turett)

Let $(\gamma_k) \subset (0,1)$ such that $\lim_k \gamma_k = 1$. Then

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |a_n|, \qquad x = \sum_{n=1}^{\infty} a_n e_n$$

is an equivalent norm in ℓ_1 and $(\ell_1, |||\cdot|||)$ does not have an a.i.c. of ℓ_1.

Could $(\ell_1, |||\cdot|||)$ have the FPP...?

Theorem (P.K. Lin, 2008)

$(\ell_1, |||\cdot|||)$ *has the FPP.*

- $(\ell_1, |||\cdot|||)$ was the first known nonreflexive Banach space with the FPP
- The above shows that FPP does not imply reflexivity
- The above shows that the FPP is not preserved by isomorphisms: $(\ell_1, \|\cdot\|_1)$ fails the FPP whereas $(\ell_1, |||\cdot|||)$ does fulfill the FPP.

Sequentially separating norms

Definition

Let X be a Banach space with a Schauder basis. An equivalent norm $p(\cdot)$ is a sequentially separating norm (s.s.n.) if for every $\epsilon > 0$ there exists some $k \in \mathbb{N}$ such that

$$p(x) + \limsup_n p(x_n) \leq (1+\epsilon) \limsup_n p(x + x_n)$$

whenever $k \leq x$ and $(x_n)_n$ is a block basic sequence in X.

Examples of sequentially separating norms

- $\| \cdot \|_1$ and P.K. Lin's norm $||| \cdot |||$ in ℓ_1 are s.s.n.
- For Nakano sequence Banach spaces $\ell^{(p_n)}$ with $\lim_n p_n = 1$, the usual norm is s.s.n.

- Let $p_0(\cdot)$ be a premonotone and s.s.n. Define

$$p_1(x) = \sup_k \gamma_k p_0(Q_k(x)) \qquad (\gamma_k)_k \subset (0,1), \ \lim_k \gamma_k = 1.$$

  Then p_1 is a (premonotone) s.s.n.
- By recurrence: $p_n(x) = \sup_k \gamma_k p_{n-1}(Q_k(x))$ are s.s.n. for every $n \in \mathbb{N}$.

Theorem (A. Barrera-Cuevas, M.A. Japón)

Let X be a Banach space with a boundedly complete Schauder basis. Let $p_0(\cdot)$ be an equivalent premonotone sequentially separating norm.
Then (X, p_1) has the FPP and so does (X, p_n) for every $n = 1, 2, \ldots$.

Let $\mathcal{P}(X)$ denote the set of equivalent norms in a Banach space X.

Theorem

Let X be a Banach space with a boundedly complete Schauder basis. Assume there exists some $p_0 \in \mathcal{P}(X)$ which is a premonotone s.s.n.
Then for every $n \in \mathbb{N}$, the set of $\mathcal{P}(X)$ with the FPP contains n-dimensional affine manifolds.

For instance, for every $\lambda_i \geq 0$, the norm

$$p_0(\cdot) + \lambda_1 p_1(\cdot) + \cdots \lambda_n p_n(\cdot)$$

verifies the FPP.

A little bit of geometry

Which Banach spaces can be renormed with a sequentially separating norm?

Theorem

Let X be a Banach space which a Schauder basis which admits an equivalent sequentially separating norm. Then X has the Schur property and it is hereditarily ℓ_1.

There exist some Banach spaces with boundedly complete Schauder basis, non-isomorphic to ℓ_1 and with sequentially separating equivalent norms.

Theorem (C. Hernández-Linares,M. Japón)

- *If G is a separable compact group, its Fourier-Stieltjes algebra $B(G)$ is FPP-renormable.*

- *There exist some nonreflexive and non-isomorphic-ℓ_1 Banach subspaces of $L_1[0,1]$ which are FPP-renormable.*

Can $L_1[0,1]$ be renormed to have the FPP?

Theorem

- *$L_1[0,1]$ can be renormed to have the FPP for affine nonexpansive mappings.*

$$|||x||| := \sup_{k\geq 1}\left(1-\frac{1}{2k}\right)\frac{1}{k}f^{**}\left(\frac{1}{k}\right),$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$ for $t>0$ and f^* denotes the decreasing rearrangement of f.

- *If $\mathcal{M}$ a finite von Neumann algebra, $L_1(\mathcal{M})$ can be renormed to have the FPP for affine nonexpansive mappings*

Some of the main references

- K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.
- W.A. Kirk, B. Sims. Handbook of Metric Fixed Point Theory. Kluwer Acad. Publ., Dordrecht, 2001.
- P. K. Lin, *There is an equivalent norm on ℓ_1 that has the fixed point property*. Nonlinear Anal., 68 (8) (2008), 2303-2308.
- P. N. Dowling, C.J Lennard and B. Turett, *The fixed point property for subsets of some classical Banach spaces*, Nonlinear Anal. **49**, 141-145, 2002.
- T. Domínguez Benavides, *A renorming of some nonseparable Banach spaces with the fixed point property*, J. Math. Anal. Appl., 350(2) (2009), 525-530.
- C. A. Hernández-Linares and M. A. Japón. *A renorming in some Banach spaces with applications to fixed point theory*. J. Funct. Anal. 258 (2010), 3452-3468.
- C. A. Hernández-Linares, M. A. Japón, E. Llorens-Fuster. *On the structure of the set of equivalent norms on ℓ_1 with the fixed point property*. J. Math. Anal. App. 387 (2012), 645-54.
- C. A. Hernández-Linares, M. A. Japón. *Rays of equivalent norms with the fixed point property in some nonreflexive Banach spaces*. J. Nonlinear Convex Anal. 15, n. 2, (2014), 355-377.
- A. Barrera-Cuevas, M.A. Japón. *New families of non-reflexive Banach spaces with the fixed point property*. Preprint.