Topologies on spaces of holomorphic functions

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Problem

To find Banach spaces E and open subsets $U \subset E$ such that $\tau_\omega = \tau_\delta$ on $H(U)$.

Definition (Nachbin)

A seminorm p on $H(U)$ is τ_ω continuous if there is a compact subset $K \subset U$ with the following property:
If V is open and $K \subset V \subset U$, then there is $C > 0$ such that

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Definition (Coeuré, Nachbin)

A seminorm p on $H(U)$ is τ_δ continuous if for each sequence $(V_n)_{n=1}^{\infty}$ of open subsets of U such that

$$V_1 \subset V_2 \subset V_3 \subset \cdots \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} V_n = U$$

there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that

$$p(f) \leq C \sup_{x \in V_{n_0}} |f(x)| \qquad \forall f \in H(U).$$

Theorem (Dineen)

Let E be a Banach space with an unconditional Schauder basis. If U is a balanced open subset of E, then $\tau_\omega = \tau_\delta$ on $H(U)$.

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Theorem (Dineen, Mujica)

Let E be a separable Banach space with the bounded approximation property.
If U is a balanced open subset of E, then $\tau_\omega = \tau_\delta$ on $H(U)$.

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Problem

Let U and V be open subsets of a Banach space E.

$$\tau_\omega = \tau_\delta \text{ on } H(U) \stackrel{?}{\Longrightarrow} \tau_\omega = \tau_\delta \text{ on } H(V).$$

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Let U be a balanced open subset of a Banach space E.
Let A be a closed bounded subset of E such that $A \subset U$.
$\tau_\omega = \tau_\delta$ on $H(U \backslash A)$?

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Theorem (Hartogs)

Let U be an open subset of $\mathbb{C}^n$, $n \geq 2$.
Let K be a compact subset of U such that $U \backslash K$ is connected.
If $f \in H(U \backslash K)$, then there is $\widetilde{f} \in H(U)$ such that $f = \widetilde{f}$ on $U \backslash K$.

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Theorem

Let U be a balanced open subset of a Banach space E, $dim(E) \geq 2$.
Let A be a closed bounded subset of U such that $U \backslash A$ is connected.
If $f \in H(U \backslash A)$, then there is $\widetilde{f} \in H(U)$ such that $f = \widetilde{f}$ on $U \backslash A$.

Theorem (Coeuré, Hirschowitz)

Let $V \subset U$ be connected open subsets of a Banach space. If every $f \in H(V)$ has an extension $\widetilde{f} \in H(U)$, then

$$f \in (H(V), \tau_\delta) \mapsto \widetilde{f} \in (H(U), \tau_\delta)$$

is a topological isomorphism.

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Theorem (Josefson)

Let I be an uncountable set.
There are open subsets $V \subset U \subset c_0(I)$ such that
every $f \in H(V)$ has an extension $\widetilde{f} \in H(U)$ but the mapping

$$f \in (H(V), \tau_\omega) \mapsto \widetilde{f} \in (H(U), \tau_\omega)$$

is not continuous.

Theorem

Let E be a separable Banach space with the bounded approximation property.
If U is a balanced open subset of E, A is a closed bounded subset of U and $U \backslash A$ is connected, then $\tau_{\omega} = \tau_{\delta}$ on $H(U \backslash A)$.

Proof

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Dineen: as $U \subset Spec(H(V), \tau_\omega)$, the mapping

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is a topological isomorphism.

Dineen, Mujica: $\tau_\omega = \tau_\delta$ on $H(U)$, so $\tau_\omega = \tau_\delta$ on $H(V)$.

Theorem
Let A be a closed bounded subset of a Banach space E such that $E \backslash A$ is connected.
Then $\tau_\omega = \tau_\delta$ on $H(E)$ if and only if $\tau_\omega = \tau_\delta$ on $H(E \backslash A)$.

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Theorem

If A is a closed bounded subset of ℓ_∞ and $\ell_\infty \backslash A$ is connected,
then $\tau_\omega < \tau_\delta$ on $H(\ell_\infty \backslash A)$.

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