Tauberian Polynomials

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XIII Encuentros Murcia-Valencia

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Davie and Gamelin showed (10 years latter) that $\|\tilde{P}\| = \|P\|$.

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- X non reflexive and $P : X \to Y$ weakly compact N-homogeneous polynomial $\Rightarrow P$ is not Tauberian. In particular, every element of $\mathcal{P}_{wu}(^N X, Y)$ is not a Tauberian polynomial.

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More explicitly, each $P \in \mathcal{P}(^N X, Y)$ can be identified with a linear operator $L_P \in \mathcal{L}(\widehat{\otimes}_{N,s,\pi} X, Y)$ such that $P(x) = L_P(x \otimes \cdots \otimes x) \;\; \forall x \in X$.

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If $\delta : X \to \widehat{\otimes}_{N,s,\pi} X$ is the N-homogeneous polynomial given by $\delta(x) = x \otimes \cdots \otimes x$, we have $P = L_P \circ \delta$ for every $P \in \mathcal{P}(^N X, Y)$.

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Assume that $P \in \mathcal{P}(^N X, Y)$ is a Tauberian polynomial and $T \in L(Y, Z)$ is a Tauberian operator, then $T \circ P$ is a Tauberian polynomial. A partial converse holds: if $T \circ P$ is a Tauberian polynomial, then P itself is Tauberian.

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However:

$P =$ Tauberian polynomial $\not\Rightarrow$ its linearization T_P is Tauberian.

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$\ell_2 \widehat{\otimes}_{s,\pi} \ell_2$ contains a copy of ℓ_1 $\Rightarrow$ the quotient mapping

$$q : \ell_2 \widehat{\otimes}_{s,\pi} \ell_2 \to \frac{\ell_2 \widehat{\otimes}_{s,\pi} \ell_2}{\ell_1}$$

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However, the polynomial

$$x \in \ell_2 \mapsto P(x) := x \otimes x + \ell_1 \in \frac{\ell_2\widehat{\bigotimes}_{s,\pi}\ell_2}{\ell_1}$$

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$P(x) = q(x \otimes x)$ $\Rightarrow q$ is the linearization of P.

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In $X = C([0,1])$, the identity operator $I : X \to X$ is Tauberian (clear), but $P : X \to X$ defined by $P(f) = I(f)^2 = f^2$ for all $f \in X = C([0,1])$ is not Tauberian.

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Steps:
• Associate the function $g_0 := \chi_{[0,1[} - \chi_{\{1\}}$, which is bounded and Borel measurable on $[0,1]$, to an element of $C([0,1])'' \backslash C([0,1])$.

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From this $g_0 \in C([0,1])'' \backslash C([0,1])$ and $\tilde{P}(g_0) \in C([0,1])$ since g_0^2 is the constant function 1 which is in X.

Tauberian Operators (Known characterizations)

Proposition

If $T : X \to Y$ is linear and continuous, the following are equivalent:

(1) $T''(X'' \setminus X) \subset Y'' \setminus Y$ (i.e., T is Tauberian).

(2) If $B \subset X$ is a bounded set such that $T(B)$ is weakly relatively compact, then B is weakly relatively compact.

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Proposition

If X is a weakly sequentially complete Banach space, Y is an arbitrary Banach space and $T : X \to Y$ is linear and continuous, the following are equivalents:

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Consequently, the weak topology can not play the same role as in the linear setting. For instance, the equality $T^{tt}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}$ for all bounded and convex subset A of X cannot be extended to the case of polynomials since in general $P(A)$ is not convex.

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The **polynomial topology** τ_p (resp. τ_{p^N}) on X is the smallest topology for which a net (x_α) converges to x if and only if $P(x_\alpha) \to P(x) \quad \forall P \in \mathcal{P}(X)$ (resp. $\forall P \in \mathcal{P}(^m X)$ para todo $m \leq N \in \mathbb{N}$).

Tauberian Polynomials

Because of the lack of linearity, some difficulties arise:

• Given $P \in \mathcal{P}(^N X, Y)$, we can't say in general that $P(X)$ is a linear subspace of Y.

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Consequently, the weak topology can not play the same role as in the linear setting. For instance, the equality $T^{tt}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}$ for all bounded and convex subset A of X cannot be extended to the case of polynomials since in general $P(A)$ is not convex.

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The **polynomial-star topology** τ_{p^*} (respectively, $\tau_{p^{*N}}$) on X'' is the smallest topology for which a net (z_α) converges to z if and only if $\tilde{P}(z_\alpha) \to \tilde{P}(z) \quad \forall P \in \mathcal{P}(X)$ (resp. $\forall P \in \mathcal{P}(^m X)$ para todo $m \leq N \in \mathbb{N}$).

The polynomial topologies τ_p and τ_{p^*}

Theorem (Davie and Gamelin)

Let S be a bounded subset of X and suppose that $z \in X''$ is w^*-adherent to S. Then there exists a net (x_α) in X such that each x_α is an arithmetic mean of distinct elements of S, and $P(x_\alpha)$ converges to $\tilde{P}(z)$ for all $P \in \mathcal{P}(X)$.

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There is not a general Banach Alaoglu Polynomial Theorem, but we showed the following:

Banach Alaoglu Polynomial Theorem

If X is a Banach space, then $\mathcal{P}_{wu}(^N X) = \mathcal{P}(^N X)$ if and only if every bounded and $\tau_{p^{*N}}$-closed subset A of X'' is compact in the $\tau_{p^{*N}}$-topology. A similar statement holds for the equality $\mathcal{P}(X) = \mathcal{P}_{wu}(X)$ and the τ_{p^*}-topology.

Theorem

Let $y \in Y$, and $P \in \mathcal{P}(^N X, Y)$. The following statements are equivalent:

(a) $\tilde{P}^{-1}(y) \subset X$.

(b) If $(x_\alpha)_{\alpha \in \Lambda}$ is a net in X such that $P(x_\alpha) \xrightarrow{w} y$, then every $\tau_{p^{*N}}$-cluster point of $\left(x_\alpha\right)_{\alpha \in \Lambda}$ belongs to X.

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Corollary

Let $P \in \mathcal{P}(^k X, Y)$. The following statements are equivalent:

(a) $\tilde{P}^{-1}(0) = P^{-1}(0)$.

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Tauberian Polynomials

Theorem

Let $P \in \mathcal{P}(^N X, Y)$. Consider the following statements:

(a) P is Tauberian.

(b) $P(B_X)$ is weakly closed and $x'' \in X$ whenever $x \in X, x'' \in X''$ and $\tilde{P}(x'') = P(x)$.

(c) $P(B_X)$ is weakly closed and $\tilde{P}^{-1}(0) = P^{-1}(0)$.

Then $(b) \Rightarrow (a)$ and (c), and whenever $\mathcal{P}_{wu}(^N X) = \mathcal{P}(^N X)$, $(a) \Rightarrow (b)$.

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$$P(x) = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n^2, \text{ for all } x \in \ell_2 .$$

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But $P(B_{\ell_2})$ is not closed in $\mathbb{C}$ since $\left(P(e_n)\right) = \left(1 - \frac{1}{n}\right) \to 1$ and $1 \notin P(B_{\ell_2})$ as $\left|\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n^2\right| < 1 \ \ \forall x \in B_{\ell_2}$.

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• T is well defined and Ker $T = \{0\}$ (easy).

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- Now show that $T^{tt} : \ell_\infty \to \ell_2$ is given by $T^{tt}((y_n)) = (z_n)$ where

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Hence

$$P^{-1}(0) = \text{Ker } T = \{0\} = \text{Ker } T^{tt} = \tilde{P}^{-1}(0).$$

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Finally take

$$x_0 = \left(\frac{1}{2^{n-1}}\right)_{n=1}^{\infty} \in c_0 \ \text{ and } \ y_0 = \left(-\sum_{k=1}^{n} \frac{1}{2^{k-1}}\right)_{n=1}^{\infty} \in \ell_\infty \setminus c_0 \ ,$$

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Hence $\tilde{P}(y_0) = P(x_0)$ for some $x_0 \in c_0$ and $y_0 \in \ell_\infty \backslash c_0$, as we wanted.

Tauberian Polynomials

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So, P is Tauberian (since ℓ_2 is reflexive) and $P(A) \subset \mathbb{C}$ is weakly relatively compact (clear) and $A \subset \ell_2$ is not τ_{p^N}-relatively compact.

Tauberian Polynomials

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Theorem

Let $P \in \mathcal{P}(^N X, Y)$ be a Tauberian polynomial. If C is a bounded subset of X such that $P(C)$ is weakly relatively compact, then the closure of C for the $\tau_{p^{*N}}$-topology lies in X. If moreover, $\mathcal{P}(^N X) = \mathcal{P}_{wu}(^N X)$, then C is τ_{p^N}-relatively compact.

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For separable Banach spaces we have

Theorem

Let X be separable and $\mathcal{P}(^N X) = \mathcal{P}_{wu}(^N X)$. The polynomial $P \in \mathcal{P}(^N X, Y)$ is Tauberian if and only if every bounded subset C of X such that $P(C)$ is weakly relatively compact is weakly relatively compact.

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Theorem

Let $P \in \mathcal{P}(^N X, Y)$. Assume that the weak topology on $\overline{P(B_X)}^w$ is metrizable. If every bounded subset C of X such that $P(C)$ is weakly relatively compact is τ_{p^N}-relatively compact, then P is Tauberian and $P(B_X)$ is weakly closed.

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