Gustavo Adolfo Muñoz Fernández Homenaje al Prof. Richard M. Aron Universidad de Valencia

Joint work with

J. L. Gámez, D. Pellegrino y J. B. Seoane

Valencia, 13 de diciembre 2014



Gustavo Adolfo Muñoz Fernández On a characterization of continuity

A characterization of continuity

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous then

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- If $f : \mathbb{R} \to \mathbb{R}$ is a function such that
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then f is necessarily continuous.

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A characterization of continuity

Motivation for the generalization

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Motivation for the generalization

• A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in \mathbb{R} if and only if

 $f^{-1}(U)$ is open for all open set $U \subset \mathbb{R}$.

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• The answer again is no, but the result is highly nontrivial.

Theorem (Velleman (1997))

There are not families ${\mathcal F}$ and ${\mathcal G}$ of subsets of ${\mathbb R}$ such that

 $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f(U) \in \mathcal{G}$ for all $U \in \mathcal{F}$.

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Theorem (Velleman (1997), Hamlett (1975), White (1968))

There are two families \mathcal{F} and \mathcal{G} of subsets of \mathbb{R} such that $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if

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 $f:\mathbb{R} o \mathbb{R}$ is continuous if and only if

1
$$f(U) \in \mathcal{F}$$
 for all $U \in \mathcal{F}$, and

2 $f(V) \in \mathcal{G}$ for all $V \in \mathcal{G}$.

The characterization again

A plausible choice for ${\mathcal F}$ and ${\mathcal G}$ in the previous theorem is the following:

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A generalization of the characterization

- O The same result holds for functions f : X → Y where X is first countable and locally connected and Y is regular.
- However the result is not true for functions between metric spaces in general.

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Derivatives as connected functions

Theorem (Darboux)

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is a Darboux functions, i.e., f' transforms intervals into intervals.

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Derivatives are not necessarily continuous

The derivative of

$$G(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0.

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Derivative with uncountably many discontinuities

Volterra construction

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Derivative with uncountably many discontinuities

Volterra construction

• Choose
$$x_0 > 0$$
 so that $G'(x_0) = 0$.

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Volterra construction

- Choose $x_0 > 0$ so that $G'(x_0) = 0$.
- **2** Define $G_0 : (0, 2x_0) \rightarrow \mathbb{R}$ as follows:

$$G_0(x) = egin{cases} G(x) & ext{if } x \in (0, x_0], \ G(2x_0 - x) & ext{if } x \in [x_0, 2x_0) \end{cases}$$

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Using translations and homothetic transformations of G₀, F coincides with a copy of G₀ in every interval (a, b) of [0, 1] \ C, where C is the Cantor set.

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• Using translations and homothetic transformations of G_0 , F coincides with a copy of G_0 in every interval (a, b) of $[0,1] \setminus C$, where C is the Cantor set.

• We put
$$F(x) = 0$$
 for all $x \in C$.

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• We put
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 for all $x \in C$.

• F is differentiable in [0,1] but F' is not continuous in C.

Derivative with uncountably many discontinuities



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Derivatives that are discontinuous almost everywhere

Definition (Aron, Gurariy, and Seoane (2004))

A subset V of a linear space E is λ -lineable if $V \cup \{0\}$ contains a linear space of dimension λ .

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The set of differentiable functions on \mathbb{R} whose derivatives are discontinuous almost everywhere is c-lineable.
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The set of differentiable functions on $\mathbb R$ whose derivatives are discontinuous almost everywhere is $\mathfrak c\text{-lineable}.$

Corollary

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous almost everywhere is \mathfrak{c} -lineable.

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Everywhere surjective functions

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is everywhere surjective if $f(I) = \mathbb{R}$ for all nontrivial interval I.

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Corollary

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous everywhere is 2^c-lineable.

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Functions that transform compact sets into compact sets

Theorem (Gámez, Muñoz, and Seoane (2011))

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that have finite range (and hence transform any set into a compact set) and are everywhere discontinuous is 2^c-lineable.

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Sketch of proof

• Let *H* be a Hamel basis of \mathbb{R} over \mathbb{Q} .

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- Let *H* be a Hamel basis of \mathbb{R} over \mathbb{Q} .
- Let $\varphi : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ a \mathbb{Q} -linear isomorphism.

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- For all $A \subset H$ we define $f_A(x) := \chi_{([A] \setminus \{0\})^{\mathbb{N}}}(\varphi(x))$, for all $x \in \mathbb{R}$.

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- Let $\varphi : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ a \mathbb{Q} -linear isomorphism.
- For all $A \subset H$ we define $f_A(x) := \chi_{([A] \setminus \{0\})^{\mathbb{N}}}(\varphi(x))$, for all $x \in \mathbb{R}$.
- Choose $h_0 \in H$ and consider $F = \{f_A : \emptyset \neq A \in \mathcal{P}(H), h_0 \notin A\}$. Then F is linearly independent and its cardinality is 2^{c} .

A characterization of continuity for polynomials

Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011)) If E is a normed space and P is a polynomial on E then P is continuous if and only it transforms compact sets into compact sets.

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If *E* is a normed space and $P \in \mathcal{P}(^{n}E)$ with n = 1, 2, then *P* is continuous if and only it transforms connected sets into connected sets.

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If *E* is a normed space and $P \in \mathcal{P}(^{n}E)$ with n = 1, 2, then *P* is continuous if and only it transforms connected sets into connected sets.

Conjecture

A polynomial P on a normed space E is continuous if and only if it transforms connected sets into connected sets.

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A characterization of continuity for multilinear forms

Corollary (Gámez, Muñoz, Pellegrino, and Seoane (2011))

An *n*-linear form *L* on a normed space *E* is continuous if and only if it transforms connected set in E^n into connected sets in \mathbb{R} .

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Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011))

If $n \in \mathbb{N}$ and E is a normed space of infinite dimension λ , then the sets of the non-bounded *n*-linear forms, the non-bounded *n*-linear symmetric forms, the *n*-homogeneous polynomials and the polynomials of degree at most *n* on E are 2^{λ} -lineable.

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My first entry in R. Aron's guest book

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