On a characterization of continuity

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Joint work with

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Valencia, 13 de diciembre 2014

A characterization of continuity

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If $f : \mathbb{R} \to \mathbb{R}$ is a function such that

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then f is necessarily continuous.

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- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in $\mathbb{R}$ if and only if

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- The answer again is no, but the result is highly nontrivial.

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Theorem (Velleman (1997))

There are not families $\mathcal{F}$ and $\mathcal{G}$ of subsets of $\mathbb{R}$ such that

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A generalization of the characterization

1. The same result holds for functions $f : X \to Y$ where X is first countable and locally connected and Y is regular.
2. However the result is not true for functions between metric spaces in general.

Derivatives as connected functions

Theorem (Darboux)

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Derivatives are not necessarily continuous

The derivative of

$$G(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0.

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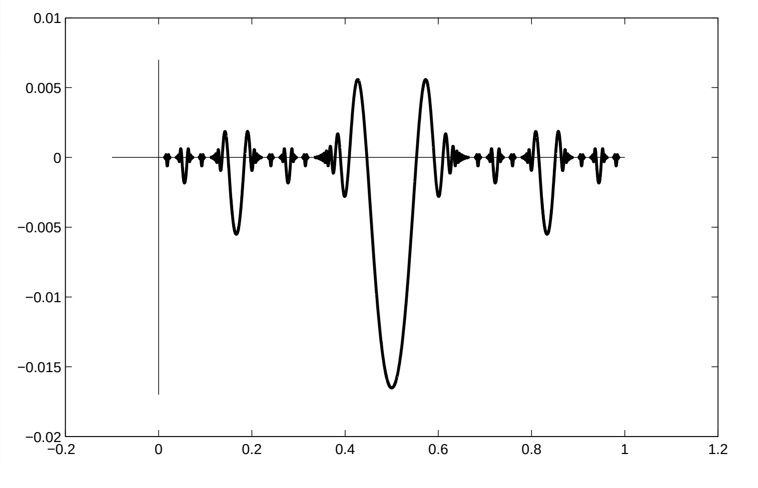
3. Using translations and homothetic transformations of G_0, F coincides with a copy of G_0 in every interval (a, b) of $[0, 1] \setminus C$, where C is the Cantor set.
4. We put $F(x) = 0$ for all $x \in C$.
5. F is differentiable in $[0, 1]$ but F' is not continuous in C.

Derivative with uncountably many discontinuities



Derivatives that are discontinuous almost everywhere

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A subset V of a linear space E is λ-lineable if $V \cup \{0\}$ contains a linear space of dimension λ.

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Corollary

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous almost everywhere is $\mathfrak{c}$-lineable.

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The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous everywhere is $2^{\mathfrak{c}}$-lineable.

Functions that transform compact sets into compact sets

Theorem (Gámez, Muñoz, and Seoane (2011))

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that have finite range (and hence transform any set into a compact set) and are everywhere discontinuous is $2^{\mathfrak{c}}$-lineable.

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- For all $A \subset H$ we define $f_A(x) := \chi_{([A]\setminus\{0\})^{\mathbb{N}}}(\varphi(x))$, for all $x \in \mathbb{R}$.
- Choose $h_0 \in H$ and consider $F = \{f_A : \varnothing \neq A \in \mathcal{P}(H),\ h_0 \notin A\}$. Then F is linearly independent and its cardinality is $2^{\mathfrak{c}}$.

A characterization of continuity for polynomials

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Conjecture

A polynomial P on a normed space E is continuous if and only if it transforms connected sets into connected sets.

A characterization of continuity for multilinear forms

Corollary (Gámez, Muñoz, Pellegrino, and Seoane (2011))

An n-linear form L on a normed space E is continuous if and only if it transforms connected set in E^n into connected sets in $\mathbb{R}$.

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Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011))

If $n \in \mathbb{N}$ and E is a normed space of infinite dimension λ, then the sets of the non-bounded n-linear forms, the non-bounded n-linear symmetric forms, the n-homogeneous polynomials and the polynomials of degree at most n on E are 2^λ-lineable.

My first entry in R. Aron's guest book