The λ-function of Aron and Lohman on Jordan structures

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Given x and y in X_1, e in $\partial_e(X_1)$, and $0 < \lambda \leq 1$, following Richard's notation, we shall say that the ordered triplet (e, y, λ) is *amenable* to x when $x = \lambda e + (1 - \lambda) y$.

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[Aron and Lohman, Pacific J. Math.'1987]

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The space X satisfies the *λ-property* if $\lambda(x) > 0$, for every $x \in X_1$, that is, for each $x \in X$ there exists a triplet (e, y, λ) amenable to x.

The Banach space X has *the uniform λ-property* when $\inf\{\lambda(x) : x \in X_1\} > 0$.

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If X is a strictly convex space, then $\lambda(x) = (1 + \|x\|)/2$ for all $x \in X_1$ and $\lambda(x)$ is attained.

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Let T be a compact metric space and let X be an infinite-dimensional strictly convex normed space. Then $C(T, X)$ has the uniform λ-property. In fact, if $x \in C(T, X)$ and $\|x\| \leq 1$, then $\lambda(x) = (1 + m)/2$, where

$$m = \inf\{\|x(t)\| : t \in T\}.$$

Moreover, if $x(t) \neq 0$ for all $t \in T$, $\lambda(x)$ is attained. If $\dim(X_{\mathbb{R}}) \geq 2$, then $C([0, 1], X)$ has the uniform λ-property.

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Let X be a strictly convex normed space. Then $\ell_\infty(X)$ has the uniform λ-property. In fact, if $x = (x_n) \in \ell_\infty(X)$, with $\|x\| \leq 1$ and $m = \inf\{\|x(n)\| : n \in \mathbb{N}\}$, then $\lambda(x) = (1 + m)/2$.

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> *"What spaces of operators have the λ-property, and what does the λ-function look like for these spaces?"*

This question motivated a whole series of papers, written between 1990 and 1997, in which Brown and Pedersen determined the exact form of the λ-function for von Neumann algebras and for unital C*-algebras.

The λ-function on C*-algebras

Let me recall that, by the Gelfand-Naimark theorem, a C*-algebra is a norm closed, self-adjoint subalgebra of some $B(H)$, the space of all bounded linear operators on a complex Hilbert space H.

The λ-function on C*-algebras

In order to describe the λ-function on a C*-algebra, Brown and Pedersen introduced the concept of quasi-invertible elements.

The λ-function on C*-algebras

[Kadison, Ann. Math.'1951]

For a C*-algebra A, the set $\partial_e(A_1)$ is precisely the set of all maximal partial isometries of A, i.e., elements $e \in A$ satisfying that ee^* and e^*e are projections and $(1 - ee^*)A(1 - e^*e) = 0$.

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The λ-function on C*-algebras

[Brown, Pedersen, J. Reine. Angew.'1995]

Let a be an element in a unital C*-algebra A, with group of invertible elements denoted by A^{-1}. The following are equivalent:

(*a*) $a \in \partial_e(A_1)A^{-1}\partial_e(A_1)$;

(*b*) There is maximal partial isometry $v \in \partial_e(A_1)$ with $\mathrm{Ker}(a) = \mathrm{Ker}(v)$, such that $a = v|a|$ and 0 is an isolated point in the spectrum, $\sigma(|a|)$, of $|a|$;

(*c*) $a \in \partial_e(A_1)A_+^{-1}$;

(*d*) There is maximal partial isometry $v \in \partial_e(A_1)$ such that a is positive and invertible in the Peirce-2 subspace $A_2(v) = vv^*Av^*v$ associated to v.

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An element a satisfying any of the above statements is called *quasi-invertible* or *Brown-Pedersen quasi-invertible* (BP-q-invertible for short).

Henceforth, the symbol A_q^{-1} will denote the set of *Brown-Pedersen quasi-invertible elements* in a unital C*-algebra A.

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[Brown, Pedersen, Math. Scand.'1997]

For every element a in the closed unit ball of a unital C*-algebra A the following formula holds:

$$\mathrm{dist}(a, \partial_e(A_1)) = \begin{cases} \max\{1 - m_q(a), \|a\| - 1\}, & \text{if } a \in A_q^{-1}; \\ \max\{1 + \alpha_q(a), \|a\| - 1\}, & \text{if } a \notin A_q^{-1}, \end{cases}$$

where $\alpha_q(a) = \mathrm{dist}(a, A_q^{-1})$ and $m_q(a) = \mathrm{dist}(a, A \backslash A_q^{-1})$.

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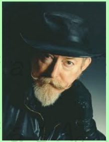
$$\operatorname{dist}(a, \partial_e(A_1)) = \begin{cases} \max\{1 - \ldots\}, & \text{if } a \ldots \\ \max\{1 + \ldots\}, & \text{if } a \ldots \end{cases}$$

L.G. Brown



G.K. Pedersen

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$$\lambda(a) = \begin{cases} \frac{1+m_q(a)}{2}, & \text{if } a \in A_1 \cap A_q^{-1}; \\ \frac{1}{2}(1-\alpha_q(a)), & \text{if } a \in A_1 \backslash A_q^{-1}, \end{cases}$$

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[Pedersen, J. Operator Theory'1991]

Every von Neumann algebra (i.e. a C*-algebra which is also a dual Banach space) satisfies the uniform λ-property, actually the expression $\lambda(a) = \frac{1+m_q(a)}{2}$ holds for every element a in the closed unit ball of a von Neumann algebra.

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In a complex Banach space X with $\dim(X) > 1$, a Riemann mapping theorem is, in general, false and a classification of simply connected domains is intractable. There are other domains which are better understood.

A domain $\mathfrak{D}$ in a complex Banach space X is symmetric if for each a in $\mathfrak{D}$ there is a biholomorphic map $S_a : \mathfrak{D} \to \mathfrak{D}$ (in Frechet's sense) with $S_a = S_a^{-1}$, such that a is an isolated fixed point of S_a.

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The open unit ball of a JB^*-triple is a bounded symmetric domain. Actually, the category of all bounded symmetric domains (with base point) is equivalent to the category of JB^*-triples.

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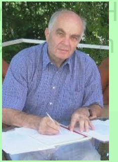

W. Kaup

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The open unit ball of a JB*-triple is a bounded symmetric domain. Actually, the category of all bounded symmetric domains (with base point) is equivalent to the category of JB*-triples.

JB*-triples

A JB*-triple is a complex Banach space, E, equipped with a continuous triple product

$$\{.,.,.\} : E \times E \times E \to E, \quad (x,y,z) \mapsto \{x,y,z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies the following axioms:

(*a*) (*Jordan Identity*)
$L(x,y)L(a,b) - L(L(x,y)a,b) = L(a,b)L(x,y) - L(a,L(y,x)b)$, for every $x,y,a,b \in E$, where $L(x,y) : E \to E$ is the linear operator defined by $L(x,y)z = \{x,y,z\}$;

(*b*) For each x in E, the operator $L(x,x)$ is hermitian with non-negative spectrum;

(*c*) $\| \{x,x,x\} \| = \|x\|^3, \forall x \in E$.

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Examples:

- Every C*-algebra, A, is a JB*-triple with respect to the triple product

$$\{x, y, z\} := \frac{xy^*z + zy^*x}{2} \tag{1}$$

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- Every complex Hilbert space is a JB*-triple.

Projections$\rightarrow$ tripotents

En element e in a JB*-triple E is called *tripotent* when $\{e, e, e\} = e$.

Each tripotent e in E induces a *Peirce decomposition* of E in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. The Peirce space $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x,e,y\}$ and involution $x^{*_e} := \{e,x,e\}$, respectively. A tripotent e in E is called complete when $E_0(e) = 0$.

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[Braun, Kaup, Upmeier, Math. Z.'1978]

For a JB*-triple E, the set $\partial_e(E_1)$ is precisely the set of all maximal/complete tripotents of E.

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[Tahlawi, Siddiqui, Jamjoom, AAA'2013, JMAA'2014]

An element a in a JB*-triple E is *Brown-Pedersen quasi-invertible* when any of the equivalent statements holds:

(*a*) There exists $b \in E$ such that the Bergmann operator $B(a, b) = Id - 2L(a, b) + Q(a)Q(b)$ is zero;

(*b*) a is von Neumann regular and its range tripotent $r(a)$ is an extreme point of the closed unit ball of E (i.e. $r(a)$ is a complete tripotent of E);

(*c*) There exists a complete tripotent $e \in E$ such that a is positive and invertible in the JB*-algebra $E_2(e)$.

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Let E be a JB*-triple. Then, for each (complete tripotent) $v \in \partial_e(E_1)$, and each element x in the closed unit ball of the Peirce-2 subspace $E_2(v)$ which is not Brown-Pedersen quasi-invertible in E we have:

$$\lambda(x) \leq \frac{1}{2}(1 - \alpha_q(x)),$$

where the symbol $\alpha_q(x)$ denotes the distance from x to the set E_q^{-1} of Brown-Pedersen quasi-invertible elements in E. Consequently, $\lambda(x) = 0$ whenever $\alpha_q(x) = 1$.

In a recent paper, we have thrown some light into the problem of determining the λ-function on the closed unit ball of a JB*-triple.

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If we set $m_q : E \to \mathbb{R}_0^+$ defined by

$$m_q(x) := \begin{cases} 0, & \text{if } x \notin E_q^{-1}; \\ \inf\{|\mu| : \mu \in \sigma_{E_2(r(x))}(x)\}, & \text{if } x \in E_q^{-1}. \end{cases}$$

[Jamjoom, Peralta, Tahlawi, Siddiqui, QJM'2014]

Let E be a JB*-triple, then

$$m_q(a) = \text{dist}(a, E \backslash E_q^{-1}),$$

for every $a \in E$. In particular, $m_q(a) = \text{dist}(a, E \backslash E_q^{-1}) = (\gamma^q(a))^{\frac{1}{2}}$, for every $a \in E_q^{-1}$, where $\gamma^q(a)$ is the quadratic conorm of a introduced and developed by Burgos, Kaidi, Morales, Pe. and Ramírez.

In a recent paper, we have thrown some light into the problem of determining the λ-function on the closed unit ball of a JB*-triple.

[Jamjoom, Peralta, Tahlawi, Siddiqui, QJM'2014]

Let a be a BP-quasi-invertible element in the closed unit ball of a JB*-triple E. Then for every $\lambda \in [\frac{1}{2}, \frac{1+m_q(a)}{2}]$ there exist e, u in $\partial_e(E_1)$ satisfying

$$a = \lambda e + (1-\lambda)u.$$

When $1 \geq \lambda > \frac{1+m_q(a)}{2}$ such a convex decomposition cannot be obtained.
Consequently, $\lambda(a) = \frac{1+m_q(a)}{2}$, for every $a \in E_q^{-1} \cap E_1$.
If $\partial_e(E_1) \neq \emptyset$, then

$$\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a)),$$

for every $a \in E_1 \backslash E_q^{-1}$.

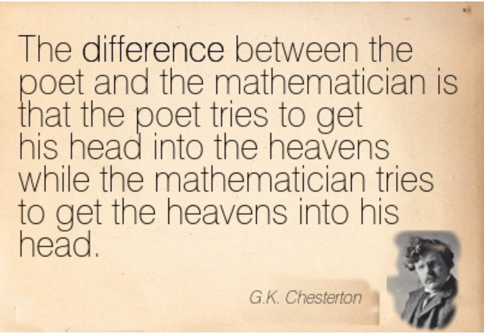
In a recent paper, we have thrown some light into the problem of determining the λ-function on the closed unit ball of a JB*-triple.

[Jamjoom, Peralta, Tahlawi, Siddiqui, QJM'2014]

Every JBW*-triple W (i.e. a JB*-triple which is also a dual Banach space) satisfies the uniform λ-property. Furthermore, the λ-function on W_1 is given by the expression:

$$\lambda(a) = \begin{cases} \frac{1+m_q(a)}{2}, & \text{if } a \in W_1 \cap W_q^{-1} \\ \frac{1}{2}(1-\alpha_q(a)) = \frac{1}{2}, & \text{if } a \in W_1 \backslash W_q^{-1}. \end{cases}$$

I finish with a motto:



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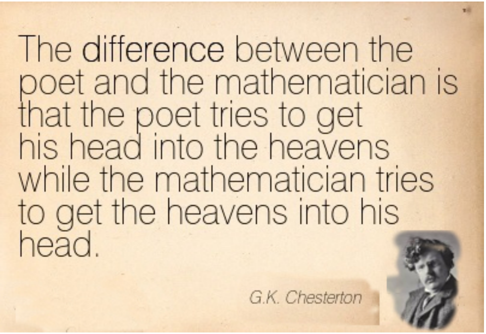

Richard, many thanks for your huge contribution to our particular heavens.