Universidad de Murcia
Departamento Matemáticas

One-side James Theorem

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Notation

$(E, \|\cdot\|)$ Banach space.

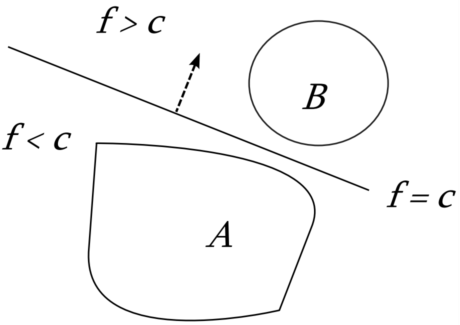
X nonempty set. If $f \in \mathbb{R}^X$ we write

$$\sup(f, X) := \sup\{f(x) \colon x \in X\}$$

$$\inf(f, X) := \inf\{f(x) \colon x \in X\}$$

We say $\sup(f, X)$ is **attained** if there is $x \in X$ with $\sup(f, X) = f(x)$.

$$\sup(f, A) < c < \inf(f, B)$$



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James Theorem

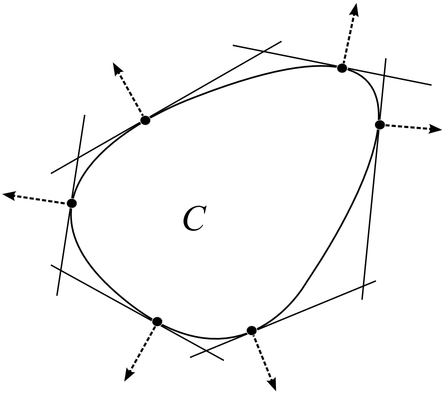
Theorem (James,1964)

If every $x^* \in E^*$ is norm-attaining, then E is reflexive.

$C \subset E$ bounded, closed, convex

Theorem (James,1964)

If every $x^* \in E^*$ attains its supremum on C, then C is weakly compact.



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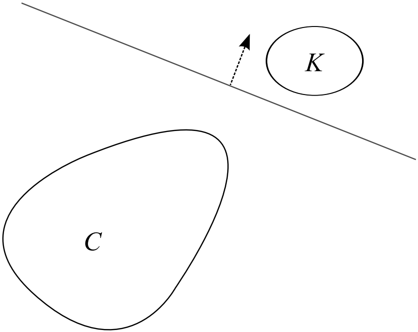
One-side James' Theorem

$C \subset E$ convex closed bounded
$K \subset E$ convex weakly compact
$C \cap K = \emptyset$

$x^* \in E^*$ with

$$\sup(x^*, C) < \inf(x^*, K)$$



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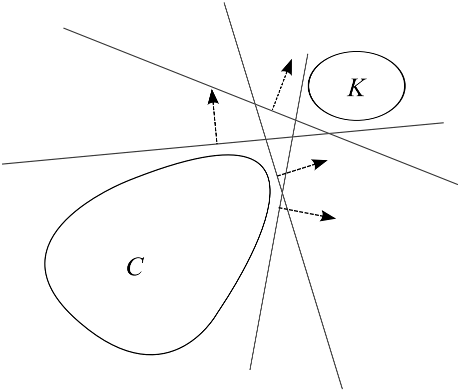

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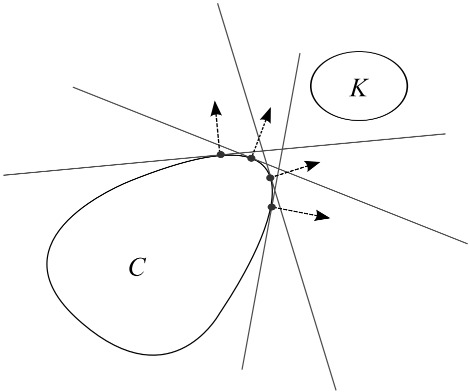

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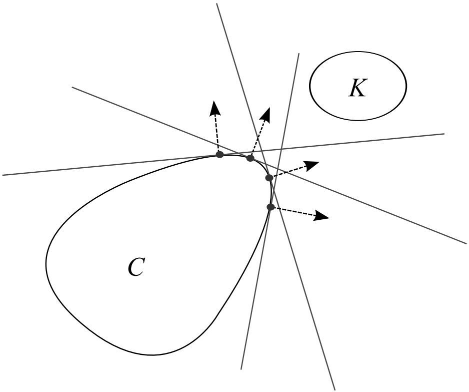
Technical hypothesis:
(B_{E^*}, ω^*) convex block compact.



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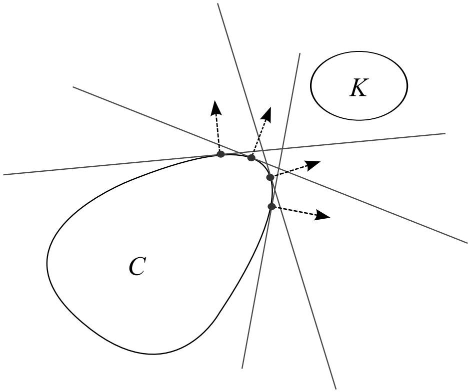
Thesis: C is weakly compact.



Motivation: Delbaen's Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A be a bounded convex and closed subset of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \notin A$. Assume that for every $Y \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\inf\{\mathbb{E}[X \cdot Y] : X \in A\} > 0$$

we have that this infimum is attained. Is A necessarily uniformly integrable?

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Yes

One-side James' theorem for $E = \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $C = -A$ and $K = \{0\}$.

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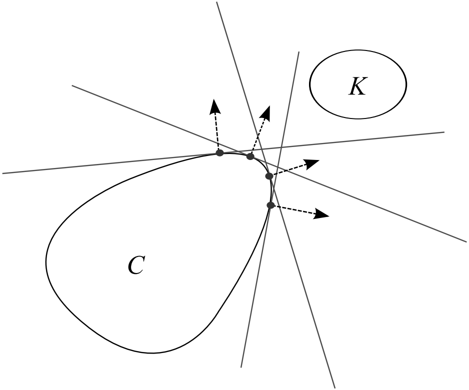


Block compactness

Let $(y_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ in (E,τ) topological vector space

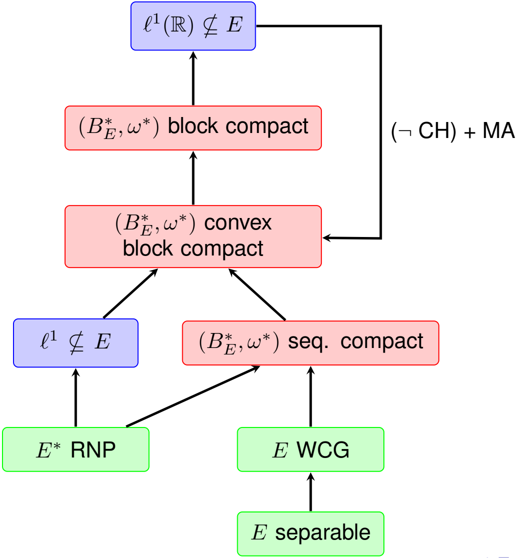
$(y_n)_{n\in\mathbb{N}}$ is a **block subsequence** of $(x_n)_{n\in\mathbb{N}}$ if there are sequences:

- $(A_n)_{n\in\mathbb{N}}$ finite subsets of $\mathbb{N}$ with $max(A_n) < min(A_{n+1})$.
- $(\lambda_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$

such that $y_n = \sum_{j\in A_n} \lambda_j x_j$ for every $n \in \mathbb{N}$.

Normalized block subsequence if $\sum_{j\in A_n} |\lambda_j| = 1$ for each $n \in \mathbb{N}$.
Convex block subsequence if $\sum_{j\in A_n} \lambda_j = 1, \lambda_j \geq 0$ for every $n \in \mathbb{N}$.

$C \subset E$ is block compact (resp. convex block compact): every sequence in C admits a normalized block subsequence (resp. convex block subsequence) which converges in C.


$\ell^1(\mathbb{R}) \not\subseteq E$
(B_E^*, ω^*) block compact
(¬ CH) + MA
(B_E^*, ω^*) convex block compact
$\ell^1 \not\subseteq E$
(B_E^*, ω^*) seq. compact
E^* RNP
E WCG
E separable

Proofs of James Theorem

- Fonf-Lindenstrauss
- Godefroy
- James
- Kalenda
- Moors
- Morillon
- Pfitzner
- Pryce
- Simons

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Sketch of Pryce's proof:

1. Suppose C is not weakly compact.
2. Find $(f_n)_n$ in E^* with some properties...
3. Find $(g_n)_n$ convex subsequence of $(f_n)_n$ with some properties...
4. If g is a weak*-cluster point of $(g_n)_n$ then

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n}(g_n - g)$$

does not attain the supremum.

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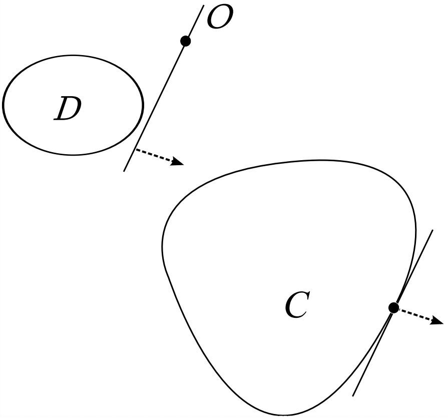
Another one-side James Theorem

$C \subset E$ convex closed bounded
$0 \notin D \subset E$ convex weakly compact

Hypothesis 1:
Every $x^* \in E^*$ with

$$\sup(x^*, D) < 0$$

attains $\sup(x^*, C)$.



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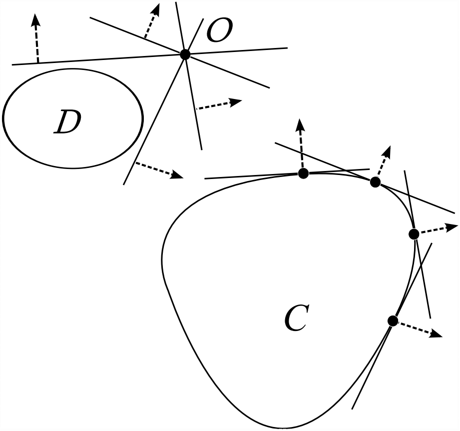

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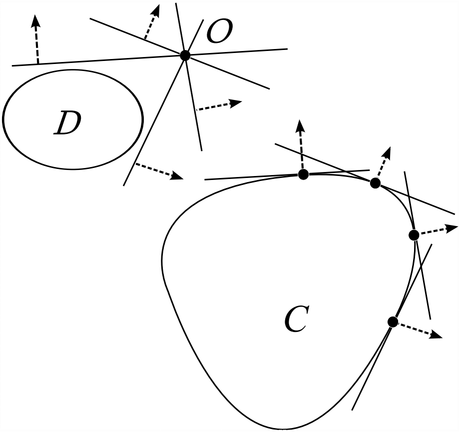
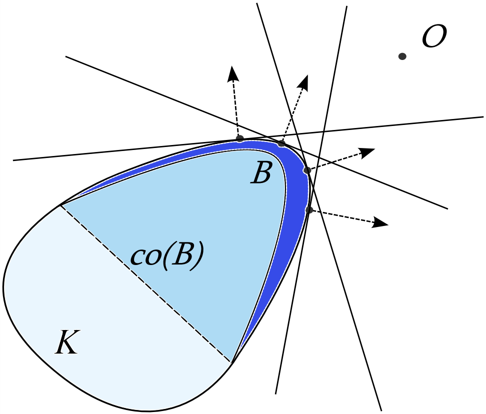


One-side Rainwater's theorem

$B \subset E^*$ bdd, $0 \notin K := \overline{\text{co}\,(B)}^{\omega^*}$

Hypothesis:
Every $x \in E$ with $\sup\,(x, K) < 0$ attains $\sup\,(x, K)$ at some $b^* \in B$.



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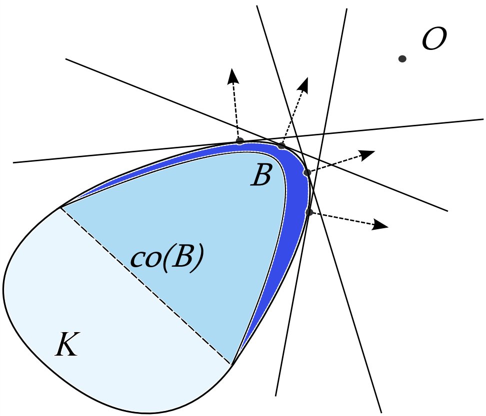
Thesis:
If $(x_n)_{n\in\mathbb{N}}$ is bounded and satisfies

$$\lim_n \langle x_n, b^* \rangle = 0 \quad \forall b^* \in B$$

then

$$\lim_n \langle x_n, x^* \rangle = 0 \quad \forall x^* \in K$$



(Unbounded) One-side Godefroy Theorem

$0 \notin D \subset E^*$ convex weak*-compact

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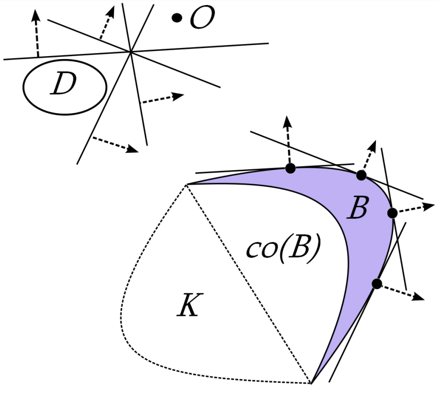

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Given $L \subset E$ bounded convex,
and $y^{**} \in \overline{L}^{\omega^*} \subset E^{**}$
there is $(y_n)_{n\in\mathbb{N}}$ in L such that
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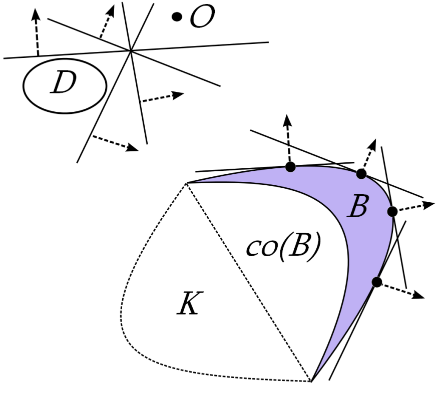

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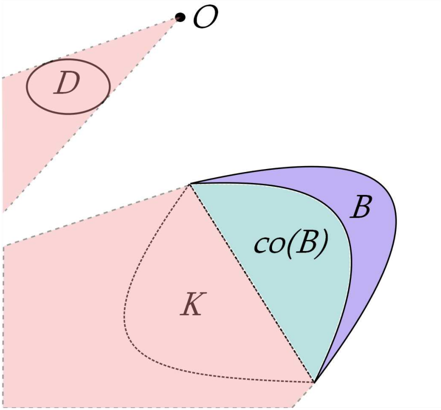
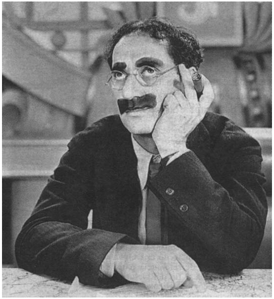
Thesis: $\overline{\mathrm{co}(B)}^{\omega^*} \subset \overline{\mathrm{co}(B) + \Lambda_D}^{\|\cdot\|}$



Things to Think



- Can we remove the “Technical hypothesis” from the one-side James theorems?
- Others one-side problems: one-side Boundary Problem, etc.
- If (B_{E^*}, ω^*) is block compact, is it in fact convex block compact?