Projective norm of products of random gaussian matrices

(C. H. Jiménez, C. González-Guillén, C. Palazuelos, I. V.)

or

Euclidean distance between Haar and gaussian matrices

(C. González-Guillén, C. Palazuelos, I. V.)

The first question

We consider a random gaussian matrix $Y = (y_{i,j})_{i,j=1}^{n}$ and a haar distributed random orthogonal matrix $U = (u_{i,j})_{i,j=1}^{n}$.

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Question 1: How many indices i,j can we choose so that all of the chosen $\sqrt{n}u_{i,j}$ are simultaneously *well approximated* by the corresponding $y_{i,j}$?

Actually a family of questions.

Studied, among others, by

Borel, 1906,
Gallardo, Stam, Yor, Diaconis, Freedman 80's
Diaconis, Eaton, Freedman, 90's
Jiang, 05

Previous results

Best estimate in variation distance (Jiang, '05)
For each $n \geq 1$, let Z_n be the $p_n \times q_n$ upper left block of a orthogonal Haar distributed random matrix Γ_n. Let G_n be the corresponding block of a random gaussian matrix.
If $p_n, q_n = o(\sqrt{n})$ then

$$\lim_n \|\mathcal{L}(\sqrt{n} Z_n) - \mathcal{L}(G_n)\| = 0.$$

Moreover, $o(\sqrt{n})$ is optimal.

Previous results

Best known estimate for the supremum in probability (Jiang, '05)

For each $n \geq 2$, let $Y_n = (y_{ij})_{i,j=1}^n$ be a random Gaussian matrix and let $U_n = (u_{i,j})_{i,j=1}^n$ be its Gram-Schmidt orthonormalization. Then U_n is Haar distributed in the orthogonal group $\mathcal{O}(n)$ and if we set

$$\epsilon_n(m) = \max_{1 \leq i \leq n, 1 \leq j \leq m} |\sqrt{n} u_{i,j} - y_{i,j}|$$

for $m = 1, 2, \cdots, n$, then $\epsilon_n(m) \to 0$ in probability as $n \to \infty$ provided $m_n = o(\frac{n}{\ln n})$ as $n \to \infty$.

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for $m = 1, 2, \cdots, n$, then $\epsilon_n(m) \to 0$ in probability as $n \to \infty$ provided $m_n = o(\frac{n}{\ln n})$ as $n \to \infty$.
Moreover, for any $\beta > 0$, we have that $\epsilon_n(\frac{n\beta}{\ln n}) \to 2\sqrt{\beta}$ in probability as $n \to \infty$.

Applications

Computational complexity of linear optics (Aaronson, Arkhipov)

Random matrices: Universal properties of eigenvectors (Tao, Vu)

Many others

Our question

What can be said of the euclidean distance of the rows of $Y - \sqrt{n}U$?

Specially in the "constant ratio regime" $\frac{m}{n} = \alpha$

Our first result

Let $m, n \in \mathbb{N}$ such that $\alpha = \dfrac{m}{n} \in (0, 1]$. Let Y, U be as above. For every $1 \leq i \leq n$, let F_i^m be the vector formed by the the first m-coordinates of the ith row of $Y - \sqrt{n}U$. Then

$$\|F_i^m\| \approx \sqrt{\left(\frac{\alpha}{2} + \frac{\alpha^2}{12} + \frac{\alpha^3}{32} + \cdots\right) m}$$

with probability exponentially close to 1.

Application I

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One uses the fact that the distance between the normalized euclidean norm and the supremum norm of a gaussian vector is $\sqrt{\ln n}$, with high probability.

Sketch of proof

The proof is very long and technical, but the tools are simple: the description of the Gram-Schmidt process and several forms of the concentration of measure phenomenom.

The second question

(One of the forms of) Grothendieck's inequality says that there exists an universal constant K_G such that for any $n \times n$ matrix $A = (a_{i,j})$

$$\sum_{i,j} a_{i,j} \langle u_i, v_j \rangle \leq K_G \sup \sum_{i,j} a_{i,j} \epsilon_i \sigma_j,$$

where $\epsilon_i, \sigma_j = \pm 1$ and $u_i, v_j \in B_{H_m}$.

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where $\epsilon_i, \sigma_j = \pm 1$ and $u_i, v_j \in B_{H_m}$.

This is equivalent to say that if we consider $\gamma = (\langle u_i, v_j \rangle)_{i,j=1}^n$ as an element of $\ell_\infty^n \otimes_\pi \ell_\infty^n$ then $\|\gamma\| \leq K_G$

The second question

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Arising from a question in Quantum Information Theory: "How many" quantum correlations are not classical?

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This leads us to study the case $\alpha = \frac{m}{n}$ constant.

The connections

The trivial one: A normalized gaussian vector $\frac{1}{(\sum_i g_i^2)^{\frac{1}{2}}}(g_1, \ldots, g_m) \approx \frac{1}{\sqrt{m}}(g_1, \ldots, g_m)$ is uniformly distributed in the sphere of H_m

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The non trivial one: A result of Ambainis et alia (essentially) says that if $\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n$ are normalized m-cuts of Haar distributed random unitary matrices, then $\|\tilde{\gamma}\| = \|(\langle \tilde{u}_i, \tilde{v}_j \rangle)_{i,j=1}^n\| > f(\alpha) > 1$ with high probability for certain range of $\alpha = \frac{m}{n}$, where f is related to the Marcenko-Pastur law.

Our second result

Let n and m be two natural numbers and $\alpha = \frac{m}{n}$. Let us consider $2n$ vectors $u_1, \cdots, u_n, v_1, \cdots, v_n$ uniformly distributed on the unit sphere of $\mathbb{R}^m$ and let us define $\gamma = (\langle u_i, v_j \rangle)_{i,j=1}^n$ (the corresponding quantum correlation matrix). We view γ as an element of $\ell_\infty^n \otimes_\pi \ell_\infty^n$.

a) If $\alpha \leq \alpha_0 \approx 0.004$ then $\|\gamma\| > 1$ with probability tending to one as n tends to infinity.

b) If $\alpha > 2$, then $\|\gamma\| \leq 1$ with probability tending to one as n tends to infinity.

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U, V are orthogonal, Haar distributed and independent of each other.
The singular values of G are distributed according to the Marcenko-Pastur law.
We consider the m biggest singular values of G, and the $n \times m$ matrices U', V', submatrices of U, V respectively, formed by the right and left singular vectors corresponding to those biggest m singular values.

Sketch of proof

$$\|\frac{n}{m}U'V'^T\|_{\ell_\infty^n \otimes_\pi \ell_\infty^n} \geq \frac{2-\epsilon}{1.6652}. \tag{1}$$

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where m is the number of singular values of G which are bigger than $(2-\epsilon)\sqrt{n}$.
For a fixed $0 < \epsilon < 2$ the Marcenko-Pastur law states that the quotient $\frac{m}{n}$ converges to the fixed number $\frac{1}{2\pi}\int_{(2-\epsilon)^2}^{4}\sqrt{\frac{4}{x}-1}dx$.

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Now we consider two independent random gaussian matrices X, Y. Their Gram-Schmidt orthonormalizations U, V are Haar distributed and, therefore, they verify Equation (1) with probability $1 - o(1)$.

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Now we consider two independent random gaussian matrices X, Y. Their Gram-Schmidt orthonormalizations U, V are Haar distributed and, therefore, they verify Equation (1) with probability $1 - o(1)$.
Let us consider now the $n \times m$ submatrices X', Y' corresponding to X, Y.

Sketch of proof

With the previous result and Grothendieck's inequality we get that, with probability $1 - o(1)$, we have

$$\|\frac{1}{m} X' Y'^T\|_{\ell_\infty^n \otimes_\pi \ell_\infty^n} \geq \frac{2-\epsilon}{1.6652} - \Big(2\big(\varphi(\alpha)\big) + \big(\varphi(\alpha)\big)^2\Big) K_G,$$

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where $\varphi(\alpha) \approx \sqrt{\left(\frac{\alpha}{2} + \frac{\alpha^2}{12} + \frac{\alpha^3}{32} + \cdots\right)}$ and K_G is Grothendieck's constant.

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where $\varphi(\alpha) \approx \sqrt{\left(\frac{\alpha}{2} + \frac{\alpha^2}{12} + \frac{\alpha^3}{32} + \cdots\right)}$ and K_G is Grothendieck's constant.

Our result follows now easily.

Thank you for your patience

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My best wishes to Richard in his 70th anniversary